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ON THE DIAMAGNETISM OF ELECTRONS IN METALS

By Francis Bitter.*

Norman Bridge Laboratory of Physics, California Institute of Technology

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In the following, it is proposed to calculate the induced diamagnetic moment due to the free electrons in a metal, using, instead of the classical assumptions of elastic collisions, the recently developed wave-nature of the electron and the Fermi statistics. The problem resolves itself into three parts: to find (1) the ψ functions for an electron in a lattice for the case of small perturbations by the atoms of the lattice, (2) the induced magnetic moment derived from the ψ functions and its dependence on the energy, (3) the total induced magnetic moment, which will then be obtained by applying the statistics for a degenerate system.

The form of the ψ functions is most easily obtained by solving the wave equation by the ordinary perturbation methods, with the potential due to the lattice playing the part of a perturbation.

$$\Delta^2\psi + \frac{8\pi^2\mu}{h^2} E\psi = \frac{8\pi^2\mu}{h^2} \epsilon V\psi. \qquad (1)$$

In (1), ϵ is a numerical constant which, for free electrons, is small. Following the usual procedure, put

$$\psi = \psi_0 + \epsilon\psi_1 \quad E = E_0 + \epsilon E_1$$

and the condition that (1) have a solution is that

$$\int (\mathrm{V} - E_1)\psi_0(l)\psi_0(k)dxdydz = 0 \qquad (2)$$

for all values of k satisfying the relation that

$$E_0(l) = E_0(k).$$

In order to proceed, it is necessary to find the form of V. For a simple cubic lattice with an elementary cell having the linear dimensions $2a$ and the origin of coördinates at the center of the cell, the following symmetry conditions must hold,

$$V(-x, y, z) = V(x, y, z)$$
$$\vdots \qquad\qquad \vdots$$
$$V(x + 2a, y, z) = V(x, y, z).$$
$$\vdots \qquad\qquad \vdots$$

We may therefore write

$$V = \sum_{pqr} B_{pqr} \cos \frac{p\pi x}{a} \cos \frac{q\pi y}{a} \cos \frac{r\pi z}{a}.$$

Any other cubic lattice may then be obtained by putting

$$V = \sum_{s} V_s$$

$$V_s = \sum_{pqr} {}_sB_{pqr} \cos \frac{p\pi(x + {}_s\delta_x a)}{a} \cos \frac{q\pi(y + {}_s\delta_y a)}{a} \cos \frac{r\pi(z + {}_s\delta_z a)}{a}.$$

The cases that will interest us particularly will be face-centered and body-centered cubic lattices. For these we may write,

Face-centered cubic	*Body-centered cubic*
${}_1\delta_x = {}_1\delta_y = {}_1\delta_z = 0$	${}_1\delta_x = {}_1\delta_y = {}_1\delta_z = 0$
${}_2\delta_x = {}_2\delta_y = 1\ {}_2\delta_z = 0$	${}_2\delta_x = {}_2\delta_y = {}_2\delta_z = 1$
${}_3\delta_x = {}_3\delta_z = 1\ {}_3\delta_y = 0$	
${}_4\delta_y = {}_4\delta_z = 1\ {}_4\delta_x = 0$	

and

$$V = \sum_{pqr} C_{pqr} \cos \frac{p\pi x}{a} \cos \frac{q\pi y}{a} \cos \frac{r\pi z}{a}. \tag{3}$$

With this form for V, it is easily shown that (2) is satisfied by

$$\psi_0(l,m,n) = A \cos \frac{l\pi x}{a} \cos \frac{m\pi y}{a} \cos \frac{n\pi z}{a}. \tag{4}$$

The inclusion of terms involving $\sin \frac{l\pi x}{a}$, etc., adds nothing to the result, so they are omitted here. The above value of ψ_0 is the "zeroth" approximation to the solution of (1), and for free electrons, where ϵ is very small, it is proposed to proceed with this approximation. The constant A is so determined that

$$\int_{-a}^{a} dx \int_{-a}^{a} dy \int_{-a}^{a} \psi_0^2 dz = 1$$
$$A^2 = \left(\frac{1}{a}\right)^3$$

The calculation of the induced moment now becomes a matter of sub-

stituting (4) into the proper formulae. It can easily be shown that in the cases under discussion the induced moment is independent of the orientation of the crystal in the magnetic field. It is sufficient, therefore, to apply the formula for the susceptibility[1,2]

$$\chi(l, m, n) = \frac{2}{3} \sum_{l'm'n'}{}' \frac{| M^{lmn}_{l'm'n'} |^2}{E(l', m', n') - E(l, m, n)} - \frac{e^2}{6mc^2} \bar{r}^2(l, m, n) \quad (5)$$

$$| M^{lmn}_{l'm'n'} |^2 = \text{const } [\int \psi_0(l', m', n') \vec{r} \times \nabla \psi_0(l, m, n) dxdydz]^2$$

$$\bar{r}^2(l_1, m, n) = \int r^2 \psi_0^2(l, m, n) dxdydz$$

$\chi(l, m, n)$ is the susceptibility per cell $8a^3$ with one electron per cell if the limits of integration in the above expression are taken as $+a$ and $-a$. The first term in (5) vanishes because it contains factors of the form

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0, \text{ etc.}$$

The second term is easily evaluated and the expression for χ becomes

$$\chi(l, m, n) = -\frac{e^2}{3mc^2} \left(\frac{\alpha}{4}\right)^2 + f(l, m, n)$$

where $f(l, m, n)$ is small compared to the constant term, and for simplicity the linear dimension of the elementary cell is taken as α instead of $2a$.

The fact that $\chi(l, m, n)$ is essentially independent of the quantum numbers, and therefore of the energy, shows that the statistics will have no influence on the total susceptibility. All electrons have the same susceptibility, and the total susceptibility is the susceptibility per electron times the number of electrons. The more rigorous procedure of evaluating $f(l, m, n)$ and applying the Fermi statistics has been carried out, and of course gives the result

$$K = -\frac{e^2}{3mc^2} \left(\frac{\alpha}{4}\right)^2 n, \quad (6)$$

where K is the volume susceptibility and n is the number of electrons per unit volume. The classical expression is identical except that instead of $\frac{\alpha}{4}$ there appears λ, the mean free path.

It is now possible to compare the formula found by Pauli[3] for the paramagnetic susceptibility K_p of the electron gas, with the above expression, henceforth called K_d. The expression found by Pauli may be written in the following form

$$K_p = \frac{1}{4\pi} \left(\frac{3}{\pi}\right)^{1/3} \frac{e^2}{mc^2} n^{1/3} \quad (7)$$

where n is the number of atoms per cc., each atom contributing one free electron. In the corresponding case, if ν is the number of atoms per cell α^3, we may write

$$\alpha^3 n = \nu$$

$$K_d = -\frac{\nu^{2/3}}{48}\frac{e^2}{mc^2}n^{1/3}. \qquad (8)$$

For body-centered lattices $\nu = 2$ and for face-centered lattices $\nu = 4$. The interesting result found is that

$$K_p/K_d = -2.4 \text{ for body-centered lattices}$$
$$= -1.5 \text{ for face-centered lattices.}$$

The results of the investigation may therefore be stated as follows. By taking into account the wave structure of the electron, a formula for the diamagnetic susceptibility of free electrons is obtained having the same form as the classical formula, but instead of the mean free path λ there appears $\frac{\alpha}{4}$ where α is the grating constant of the cubic lattice under consideration. Whereas the classical formula gave values in general much too large,[4] the new derivation gives for those substances to which it may properly be applied a value less than the paramagnetism of the free electrons. For the heavy elements K, Rb, Cs, Cu, Ag, Au, a quantitative check with experiment cannot be undertaken because K_p and K_d are both small compared to the diamagnetism of the atoms of the lattice themselves. Accurate measurements on lithium would, however, offer a reliable check on the above theory. Let K_a be the diamagnetism of the closed inner shell of the atom. K_a has been calculated by Pauling,[5] and the situation is the following:

For Lithium

K_p	K_d	K_a	
0.79	−0.33	−0.05	$\times 10^{-6}$

Here K_a is certainly small, and comparison with experiment would really check K_p and K_d rather than K_a. If this theory is correct, the volume susceptibility of lithium should be nearer to 0.4×10^{-6} than to 0.8×10^{-6}.

In conclusion, I wish to acknowledge my indebtedness to Professor P. S. Epstein for the valuable suggestions he made relating to this paper.

* NATIONAL RESEARCH FELLOW.

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ON THE EXPERIMENTAL EVIDENCE OF THE MOSAIC STRUCTURE OF BI SINGLE-CRYSTALS

BY ALEXANDER GOETZ

CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CALIFORNIA

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It is known that the description of the structure of a normal crystal as given by the theory of the atomic and molecular lattices is not complete without taking into account a certain kind of irregularities, the presence of which is dependent on the imperfection of the crystal. Thus the conception of the mosaic crystal rose, first given by Ewald and Smekal and others. In two recent papers, Zwicky[1,2] gave theoretical reasons for the assumption that the mosaic structure in a crystal is a regular one, its period depending on the character of the substance and the lattice. Thus, he obtains the aspect that a crystal has a definite structure—a kind of a secondary lattice—superposed to the molecular or atomic arrangement in the primary lattice.

The experimental facts previously known which can be interpreted in favor of this theory are not very many, and of a more or less indirect kind and mostly taken from non-metallic substances. There are only two phenomena known which may count as visible evidences of the mosaic structure. One is observed by Wood[10] on potassium chlorate crystals, explained by him as a periodical occurrence of twinning, the other concerns ultra-microscopic observations by Traube[3] of the "submicronic" state of crystals of very complex molecules during the act of solution into an almost saturated solvent. The fact that a crystal dissolves first into a large number of particles being within the visibility of an ultra-microscope before being dissolved in molecular dispersions, may be due to the presence of a mosaic block-structure.

The following is a partial report of observations of two different kinds, one dealing with microscopic measurements on Bi single-crystals, the other concerning a study of the conditions in which the orientation and perfection of a growing crystal can be predetermined by inoculation with a seed-crystal. Inasmuch as the ways in which these measurements were obtained differ considerably, the reports shall be published separately. The present paper deals with the microscopic investigation.

The material for the observations were Bi single-crystals grown by a special method (which is to be published soon) allowing one to grow metal crystals of lengths of 200–300 mm. and diameters of 2–7 mm. in *any desired orientation.* It seems necessary to emphasize this last fact for the reason that previous authors (Bridgman,[4] Tyndall,[5] Kapitza,[6] *et al*) had difficulties in growing crystals of all orientation in one and the same appa-

ratus under the same thermal conditions. The fact that it was possible by this method to grow crystals of any desired orientation shows that the method of production does not apply influential forces to the growing crystal, forces of a kind as were experienced by Kapitza (loc. cit.), Hasler[7] and the author and which will soon be discussed more extensively in another paper.

It is well known that the Bi crystal crystallizes in a hexagonal system with a rhombohedral symmetry which approaches almost a cubic one. The plane normal to the trigonal axis is a perfect cleavage plane (111) through which three other imperfect cleavage planes $(11\bar{1})$ intersect, thus forming a pseudo-octahedron. Furthermore, it is known from the measurements of Mügge[8] that any plastic deformation in a Bi crystal results in twinning along the (110) planes where the perfect cleavage interchanges its orientation with an imperfect one, due to the fact that the energy levels of both are almost the same.

With regard to the immediate occurrence* of twinning at the slightest mechanical stress applied to a crystal at normal temperatures, it is easily explainable that one observes under the microscope with small magnification (10–40×) three sets of lines crossing the main cleavage plane and intersecting each other at an angle of 60°. This observation already described by Kapitza[6] makes it easy to distinguish between the perfect and imperfect cleavage planes. Etching shows that these twin lamallae, i.e., their $(11\bar{1})$ planes, are much faster attacked by the acid than the (111) plane. Figure 1 shows the view of a slightly etched main cleavage plane which was plastically deformed, thus showing the occurrence of two different sets of twin lamellae.

If one looks at a perfect cleavage plane with a small magnification one observes besides the lamellae a formation as shown in figure 2 consisting of very irregular black regions and an innumerable amount of small curved lines. Closer investigation shows, that a cleavage on Bi never being exactly parallel to the cleavage plane, cuts a large number of layers which are distinctly separated. The irregular lines are the edges of different (111) lamellae cut in various directions, whereas the large black regions are shadows, cast by protruding edges. It is very remarkable that the cohesion of a Bi crystal parallel to the main cleavage plane is so small that one gets the impression of cutting through a parcel of very thin foils. Figure 3 shows a larger magnification of a more careful cleavage. Although the object is not etched, one can easily recognize the twinning lamella with its peculiar structure, whereas the indistorted regions show the foil-like structure of a (111) plane.

Searching for a microscopic criterium of the perfection of crystals produced by different methods, we † found that if we applied high-powered objectives (oil immersions) to a region of a very good unetched *fresh* (111) plane which was not intersected by any twinning lamella, we could observe

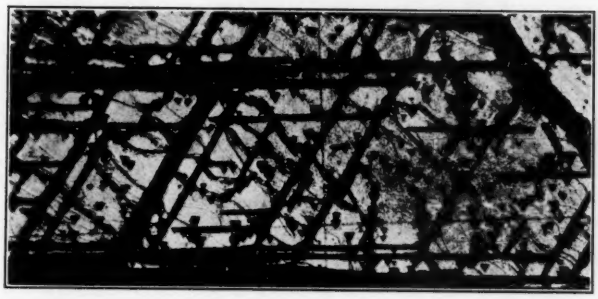

FIGURE 1

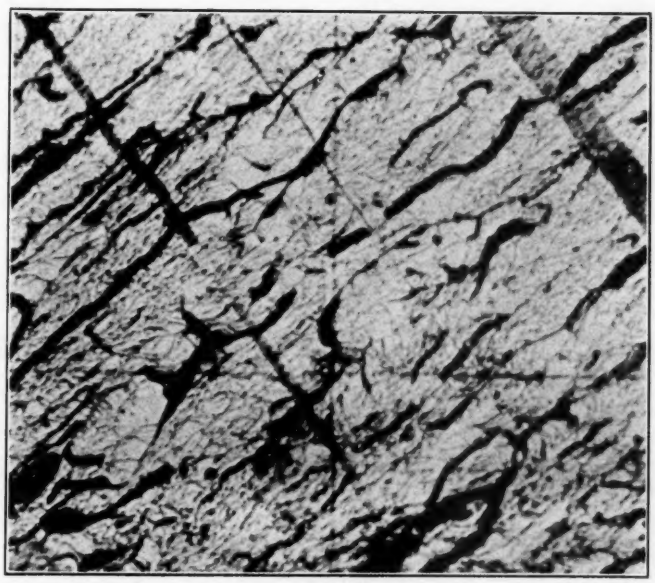

FIGURE 2

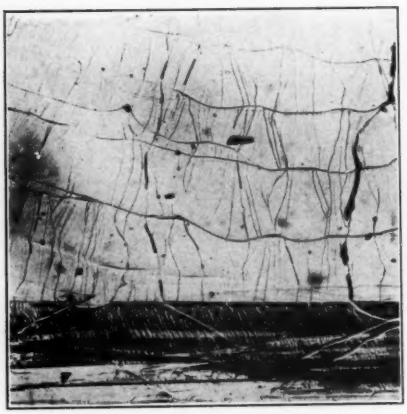

FIGURE 3

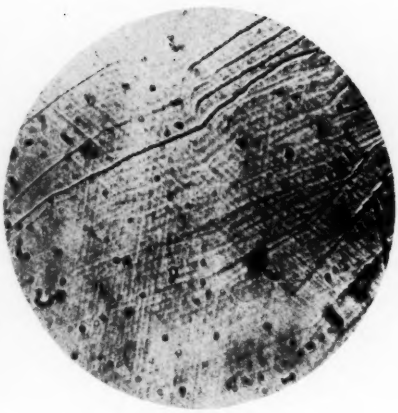

FIGURE 4

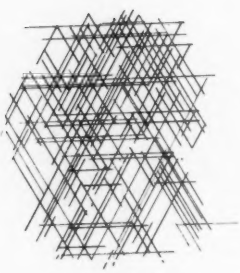

FIGURE 5

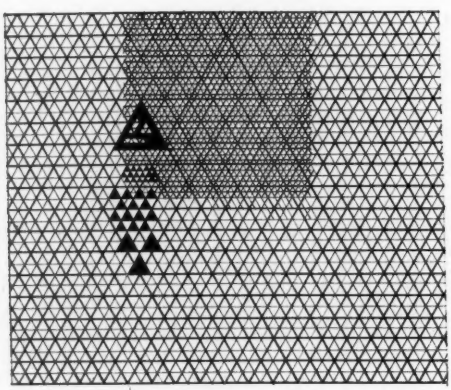

FIGURE 6

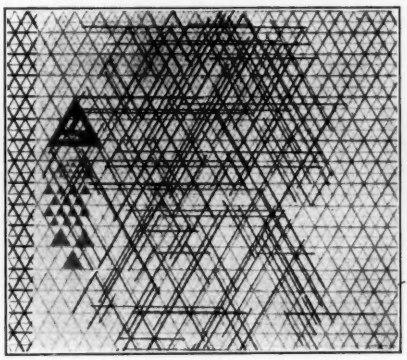

FIGURE 7

FIGURE 8

FIGURE 10

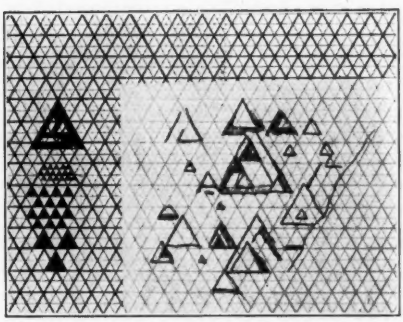

FIGURE 11

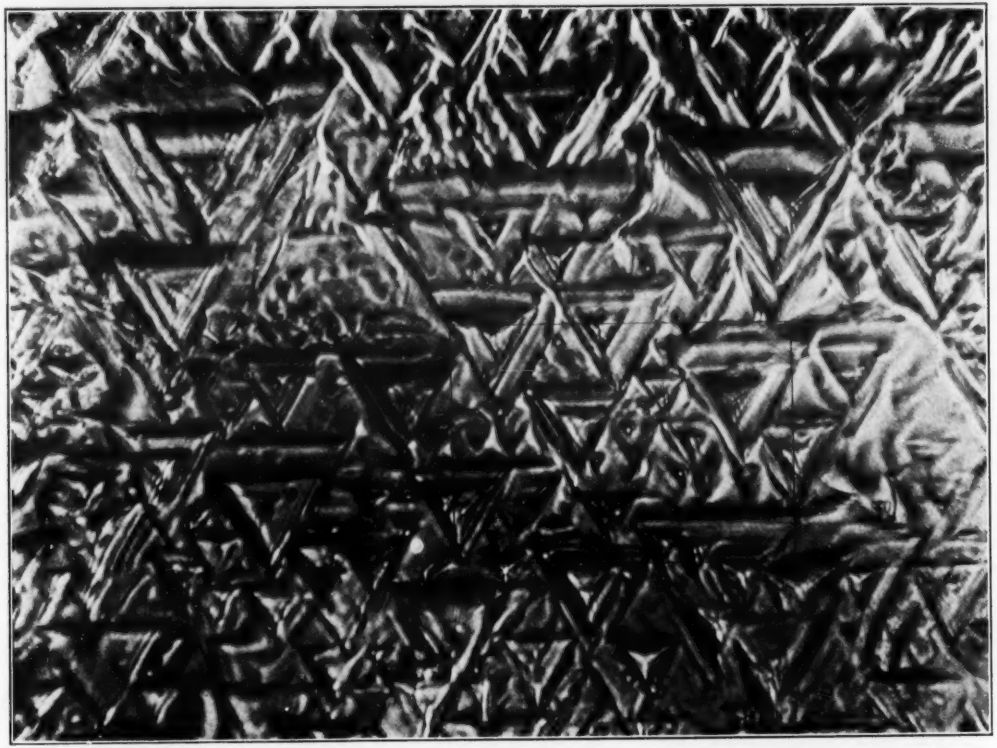

FIGURE 9

a very faint but distinctly visible pattern of lines which cross the whole field of vision. Those sets intersect each other at 60° and run parallel to the twin-lamella (in case those are perfect enough to run straight). The striking feature of these lines was their perfect equidistant occurrence which distinguishes them from the more or less rough twin lamellae.

The objective which gave the best observations was a short mount apochromatic oil immersion (Leitz) (f = 2mm.) (n.a. = 1.4) used with a Beck-Illuminator, having, for green light, a resulting resolving power of 2000 Å. The fact that the pattern appears to consist of very thin lines, the width of which is below the resolving power, makes it only possible to photograph the pattern of the shadows they cast as soon as the illumination is a trifle out of center. This can be obtained easily by bringing the microscope slightly out of focus (ca. 0.3 micron) which destroys the sharpness of the pattern but increases the contrasts so that a photograph can be taken. Although all the appliable photographic tricks were used to make the pattern more contrasting, no better success than the picture in figure 4 could be obtained.

Hence it was necessary to think of another method of reproducing and measuring the regularity and the size of the pattern to find whether or not the distances of its lines proved to be a whole-number multiple of an elementary distance. The only way to do it was by direct vision, because it was found that an especially trained eye after a sufficient time of accommodation is able to record extremely small differences in contrast. After some experimenting, it proved possible to see most of the lines when the objective was in focus. To record the lines correctly, an Abbé-drawing-apparatus was used, which allows one to bring the drawing in a direct optical relation with the original. The fainter the lines, the more regular they were, so it was sometimes necessary to bring the objective slightly out of focus, first in one direction, then in the other, thus casting faint shadows to the two sides of one line, and to mark with an artificially illuminated pencil-tip the border they had in common. Thus drawings of the type of figure 5 were obtained.

Several kinds of Bi were tried of different degrees of chemical purity and different crystalline perfection. It was found generally that *the smallest distance of the triangular pattern* over an undisturbed area on the main cleavage plane *was the same* within the limits of error, which error was estimated at not possibly more than 20% of the whole.

For measuring this distance and deciding about the existence of an integral-number-arrangement, the following method was used: The measured smallest distances were averaged and a pattern of equal sided triangles was drawn (Fig. 6) and then projected into the microscopic field of vision by means of the drawing apparatus, so that the lines of the constructed pattern fitted the visible lines on the crystal. It was observed that if the parameter

of the drawing was not exactly the same as in the specimen, the eye could separate easily the two superposed images. Finally, it was possible to obtain one drawing which disappeared almost entirely in the pattern of the specimen, thus showing that the right parameter had been obtained quite correctly. Then the crystal under the microscope was replaced by a steel-engraved micrometer of 10^{-3} cm. distance, for the calibration of the drawing. The length of the basal side of the smallest triangle was measured to be $1,4 \cdot 10^{-3}$ mm. $\pm$ 0.2 μ.

The drawn copies of the visible pattern made on transparent paper were then superimposed onto the constructed pattern as shown in figure 7 and it became evident from all obtained drawings that the visible pattern was a fragment of a pattern which fitted the constructed one, i.e., *no line was found which did not fit a line in the constructed drawing even though if not visibly connected to the coherent part of the observed pattern.* The fact that it was never possible to see the whole pattern, the constructed drawing called for, finds its explanation in the unavoidable imperfection of a cleaved plane of a soft metal; whereas the fact that there was never a line observed where there was none predicted by the drawing, leads to the conclusion that *the distances of the observed lines are whole-number multiples of a smallest-occurring unit of the stated size,* which unit is independent of the perfection, the purity, and the individuality of the Bi crystal. The question, if these observations are not due to an optical delusion caused by the resolving limit of the objective, can be answered in the negative, since the smallest resolved distance in the used arrangement is 5–6 times smaller than the size of the triangles. The power of the eyepiece was always small enough to obtain a sufficient field of vision, but large enough to separate by eye, within the distance of two elementary lines, 4 to 5 pencil lines in the projection of the drawing on the subject.

Another evidence that these triangular cells have individual qualities was derived from the etching figures obtained on these (111) planes. The etching medium giving the best pattern was found after some experimenting to be nitric acid (C. P.) diluted in distilled water (1:3) at ca. 25°. The acid has to contain a sufficient amount of bismuth nitrate to increase the speed of etching. The plane has to be cleaved just before the acid is applied. After 30 to 60 sec. the acid has to be rinsed off by warm distilled water, used for dissolving completely the traces of Bi $(NO_3)_3$ formed on the plane. The specimen should be protected after drying with a drop of immersion oil to prevent oxidation. Hence it is possible to obtain after some experience a pattern of the type shown in figure 8 giving the impression that the former plane is changed into a field of an innumerable amount of triangular pyramids of different sizes. Larger magnifications reveal that the pyramidal holes have steps of a very regular size, also that the smallest pyramids have the most perfect symmetry. The size of the average pyra-

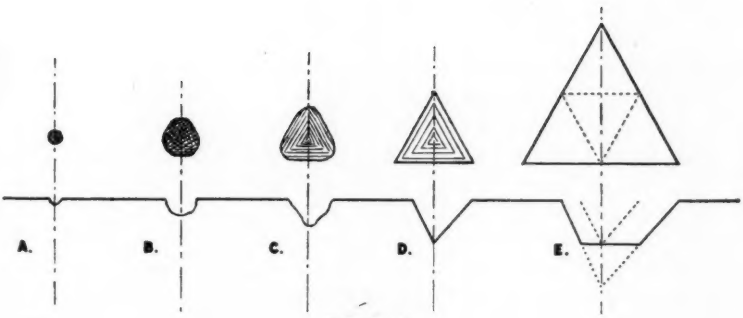

FIGURE 12

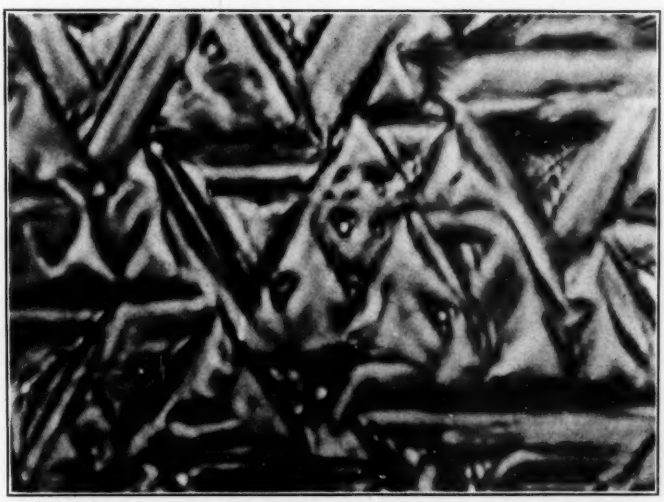

FIGURE 13

mid obtained by etching depends very much on the cleanliness of the surface, the temperature, the concentration of the acid, etc. Under the given condition (etching immediately after cleaving), the size of the pyramids is rather uniform. It was found that the size as well as the degree of uniformity depends on the time of etching. The optimum is reached in general with the time mentioned above.

A plane thus etched permits one to use large magnifications and to make a close investigation of the nature of the pyramids and of their size. (Fig. 9.)

Determining thus the qualities of the pyramids on several hundred different crystals, it was found that *there exists a "smallest" pyramid, its length of the basal plane on each side being 1.4 ± 0.2 micron, which,* within the limits of the obtained accuracy, *is the same as the size of the triangles on the unetched plane.* Furthermore, the identity of the pattern on the unetched plane and the location of the etching figure could be proved by means of making drawings first of the fresh plane (Fig. 5) and later after etching of the same region. Figure 10 shows a drawing of the etching figure and figure 11 shows that its superposition on the constructed pattern works as well as in the case of the line pattern. The fact that the etching process changes the aspect of the plane entirely makes it necessary to have an added point of reference to control the position of the object in spite of the high precision of the used mechanical device. A very small twin lamella within the microscopic field of vision fulfills this purpose admirably because of the difference of its etching pattern. Thus it could be shown that the outline of a triange after etching became the border of a pyramid, i.e., *the location of the etching pattern was predetermined by some kind of periodic inhomogeneity within the crystal.*

Special attention was paid to the study of the different phases in the genesis of the smallest etching pattern. The process in different states is shown in figure 12, which is a free-hand sketch. The dissolving process starts (in case the region of the plane is perfect) at a point which is always within the borders of a triangle, never across one line. The groove is generally round (Fig. 12 A) when its size is above the resolving power of the objective and it continues to grow in a round shape (Fig. 12 B) until it comes close to the sides of the triangle. There the dissolving process stops (Fig. 12 C) and proceeds into the corners of the triangle until a sharp-cut pyramidal pattern is reached (Fig. 12 D). If the acid is not removed at this moment, the borders of this triangle are crossed and then four pyramids unite to form one large hole (Fig. 12 E). Still later, four of these larger pyramids join, etc. But the regularity of the union of large numbers of pyramids is seldom perfect, apparently because there is only a small probability that the same surface conditions hold over larger areas.

Figure 13 shows a region indicated on figure 9 under a still larger magnifi-

cation for the demonstration of the different phases of the formation of the smallest pyramids.

The conclusion which can be drawn from these observations is that *the chemical affinity inside one block is larger in its center and considerably smaller at its borders*, which agrees very well with the prediction drawn from the theoretical aspect given by Zwicky.[2]

The fact that the side planes of the pyramids being apparently parallel to the $(11\bar{1})$ planes can form protective walls against the chemical dissolution is not in agreement with the fact that the atomic distance within these planes—as is given by Bragg diagrams—is larger than in the (111) planes (James [9]) which should result in a larger chemical affinity. The theory of Zwicky postulates that the separating walls of the "blocks" are of a higher density than indicated by the x-ray diagram, so that it is possible that the affinity of a $(11\bar{1})$ plane in the contracted state can be actually smaller than of a (111) plane.

Moreover, the same is true for the border planes of the blocks parallel to the (111) plane, which are visibly evident by the step-structure of the pyramids. Unfortunately, it is very difficult to obtain good measurements along the direction of the optical axis of the microscope at very large magnifications, hence it is not possible to make reliable statements about the depth of the steps. It must be approximately 1 micron, i.e., smaller than the basal length (Fig. 12, D and E).

The fact that the cohesive forces normal to the (111) at room temperature are very small compared with the forces in any other direction of the crystal induces the assumption, that the blocks could be easily separated parallel to this plane. One observes indeed on each cleavage along a (111) plane as mentioned above, that it cuts a large number of steps (Fig. 1). If the cleavage is done with caution, the height of these steps is very small and there is evidence that this height is the same for all of them or a whole-number-multiple and *is identical with the depth of the steps in the etched pyramids*.

Summary.—The described microscopic observations seem to lead to the conclusion that a Bi crystal consists of "blocks" of a definite size, which size is independent of the perfection of the crystal, as long as the crystal is not plastically deformed. When a deformation causes a twinning, the thickness of the smallest twin lamella is approximately the size of one "block."

The observations follow qualitatively the predictions made by the theory of Zwicky; the fact that the parameter of the "blocks" is 40% larger than the maximum given in his paper does not mean necessarily a disagreement because no quantitative statement can be made as long as the value of the Poisson number of Bi crystals is not known.

It could not be decided whether or not the described "block" is the small-

est unit of a crystal above the lattice unit. However, the existence of a smaller unit seems to be improbable for the reason that although the measured unit is 5 times above the resolving power, smaller units of the whole-number-multiple pattern could not be found, as one should expect in case of the existence of a submicroscopic unit.

Nothing can yet be said about the constitution of these "blocks" because it is impossible to investigate the nature of the lines, but it is very probable that these lines indicate regions of larger density, as indicated by their chemical behavior.

This makes us think of a "block" as an ideal crystal surrounded on all sides by a very thin hull of higher density. With regard to this structure, it is easy to identify the fissures parallel to the (111) plane in Bi crystals, indicated by the experiments of Kapitza.[6] He found that the variations of the specific resistance parallel to the trigonal axis of Bi crystals disappeared and approached the normal value as soon as he applied a pressure to the crystal in the direction of the current. There is little doubt that these fissures are caused by the above-mentioned extremely small cohesion parallel to the (111) plane along the borders of the "blocks."

I feel very much indebted to Dr. F. Zwicky for many theoretical suggestions which stimulated the investigations in this special line. Moreover, I should like to express my appreciation of the assistance given by Mr. M. F. Hasler and Mr. A. B. Focke.

* This is probably only true in the case of very pure metal and perfect single crystals.

† The discovery of this phenomenon was made practically simultaneously on the same object by Mr. Hasler, Dr. Zwicky and the author.

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THE FIRST TWO CASES OF CROSSING-OVER BETWEEN OLD-GOLD AND BULLATA FACTORS IN THE THIRD LINKAGE GROUP OF OENOTHERA[1]

BY GEORGE H. SHULL

PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY

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In a paper read before the Fifth International Genetics Congress[2] I reported the discovery of the third genetic factor (mut. *bullata*) in the third linkage group in Oenothera. This factor (b_u), which modifies nearly all the characteristics of the plant, manifests itself most conspicuously in an intense crinkling of the foliage, which first becomes manifest in quite early rosette stages. At the time the above-mentioned report was made, families which were simultaneously segregating both *bullata* and old-gold had not shown a single plant possessing both the old-gold and *bullata* characters. The ten families then available had shown 100 *bullata* plants with only yellow or sulfur flowers, while in the same family 436 non-*bullata* plants consisted of 279 yellow-flowered, 14 sulfur-flowered, 140 old-gold and 3 gold-center, thus indicating a close linkage in repulsion phase between the old-gold factor *v* and the *bullata* factor b_u.

The relation between yellow and sulfur on the one hand and that between old-gold and gold-center on the other hand is dependent upon another pair of factors *Ss* in the first linkage group. Throwing together the yellow and sulfur as non-old-gold, and the old-gold and gold-center as old-gold, these results may be summarized in the form of a simple ratio: 100 *bullata* non-old-gold : 293 non-*bullata* non-old-gold : 143 non-*bullata* old-gold, which differs from the 1:2:1 ratio only in the significantly low value of the *bullata* group, which deficiency may be plausibly explained by assuming that the *bullata* plants are inferior physiologically to the non-*bullata* plants. Assuming elimination on this account, of about 30 per cent of the *bullata* plants, the ratio would become 143:293:143, the typical 1:2:1 ratio.

The following year (1928) I had 35 families derived by self-fertilization from the doubly-heterozygous plants in the ten families on which the above-mentioned report was based. All of these 35 families repeated the results of the 1927 cultures. The total offspring showed 2250 non-*bullata* rosettes and 742 *bullata* rosettes, a ratio of 3.03:1. With respect to flower color, the parents of these cultures consisted of 32 yellow and 3 sulfur. The three families from sulfur-flowered parents consisted of 85 sulfur in the *bullata* group, while in the non-*bullata* group there were 164 sulfur-flowered and 80 gold-center, the ratio of the progenies as a whole being, thus, 85 *bullata* sulfur : 164 non-*bullata* sulfur : 80 non-*bullata*

gold-center, again almost exactly the 1:2:1 ratio. In none of these sulfur-flowered families did any plant fail to develop. The offspring of the 32 yellow-flowered diheterozygotes consisted of 495 yellow-flowered and 21 sulfur-flowered in the *bullata* group, while in the non-*bullata* group there were 1141 yellow-flowered, 53 sulfur-flowered, 502 old-gold-flowered and 11 with gold-center flowers. As the yellow and sulfur both have the *V* factor and the old-gold and gold-center both possess the *v* factor, the two groups in *bullata* can be combined and the four groups in the non-*bullata* group can be reduced to two by similar combinations. It is then seen that there are in the whole group of families 516 $b_u V$:1194 $B_u V$: 513 $B_u v$, where a 1:2:1 ratio in this number of plants would require 556:1112:556. The deficiencies in the two homozygous classes can be largely ascribed to the fact, already noted in regard to old-gold,[3] that these recessive types are to such extent inferior to the corresponding dominants that when unfavorable conditions prevent a portion of the progeny from reaching maturity, the recessive types are likely to be most affected. In support of this conclusion, we find that in the set of families now under consideration 133 *bullata* plants died in the field and 65 remained rosettes, while in the non-*bullata* class which was approximately three times as populous, only 206 died and 60 remained rosettes.

The failure to find in this large number of *bullata* and old-gold segregates grown in 1928 a single plant which had both of these recessive traits showed that the linkage between the two factors is quite close, and it was decided to lay out a special campaign for the finding of cases of crossing over. Because of the obvious rarity of crossing-over, the plan of selfing double-heterozygotes in order to get the desired recombination by the chance mating of a rare cross-over sperm with a probably like rare cross-over egg, which had been the plan of attack the year before, was abandoned, and attention was turned to the homozygous segregates. It was obvious that *bullata* segregates in the families which have been under discussion would be heterozygous for old-gold if a double-recessive cross-over gamete had taken part in their origin, and likewise that an old-gold segregate in whose beginning a double-recessive cross-over gamete had participated, would be heterozygous for *bullata*. The obvious advantage of being able to discover the segregating progenies in early rosette stages led to the choice of the old-gold segregates, as the best group to use for the finding of the desired double-recessive old-gold *bullata*. To this end 116 old-gold segregates in families that were also segregating *bullata* were selfed, and the seeds thus secured were sown in February and March, 1929, in 116 seedpans of sterilized soil. When the young seedlings began to spread themselves vigorously they were closely watched from day to day until it became quite clear whether or not any was *bullata*. Each seedpan, as soon as it could be ascertained that no *bullata* was present, was emptied

and the non-*bullata* plants were counted. Of the 116 families, 114, including 11,085 individuals, showed no *bullata* plants and the remaining 2 yielded the desired segregating progenies, the one (28114) with 115 non-*bullata* and 34 *bullata*, and the other (28133) with 78 non-*bullata* and 20 *bullata;* together these two families contained 193 non-*bullata* and 54 *bullata*, a ratio of 3.57:1.

It was with some suspense that the flowering of these plants was awaited, for it was recognized that a slip in the records by which a yellow-flowered plant had been mistakenly set down as an old-gold would have given the observed split to *bullata*, but in a family in which these *bullata* plants would all have yellow or sulfur flowers. On July 15, the first of these *bullata* plants (28133 (1)) came into bloom and proved to be the first old-gold *bullata* seen by human eyes. Later the *bullata* plants in the second of the two splitting families (28114) came into bloom, and they likewise were old-gold.

Thus was fulfilled the prediction that crossing-over would be found to occur between the old-gold and *bullata* factors. The basis of this prediction was a consideration of the experience of investigators with other organisms that multiple alleles relate to traits of closely similar nature; for example, the multiple alleles of white eye color in Drosophila are all eye-colors.

The finding of these two cases of crossing-over provides double-recessives which can be used to advantage in backcrosses, so that the adequate determination of the cross-over percentage between old-gold and *bullata* factors can be easily achieved. The occurrence of two cross-overs in the 116 pairs of gametes from which the parents of these families were produced, gives a preliminary suggestion of the cross-over value as 0.87 per cent. It will be interesting to see how nearly this value will agree with results to be secured by back-crossing of the double-heterozygotes reciprocally to the double-recessives. The necessary initial crosses have been made for these determinations.

The freedom of the old-gold and *bullata* traits from bondage to the characteristic zygote lethals of *Oe. Lamarckiana*, gives assurance that the crossing-over which is reported in this paper represents the exchange of parts between the chromosomes of a single pair. It seems probable that this chromosome pair is one of those which make up the circle of twelve chromosomes characteristic of *Oenothera Lamarckiana*. Crossing-over in this chromosome pair has already been reported between factors for old-gold and doubleness,[4] but although old-gold and *bullata* are now included in the same chromosome, and are thus in a favorable relation for linkage studies, *bullata* and *supplena* (double flowers) are still in repulsion phase. An attempt is being made to find crossing-over between *bullata* and doubleness by the same method which has proved so successful in

securing the double-recessive old-gold *bullata*, as described in this paper. Not until a crossing-over is found between *bullata* and doubleness will it be possible to study effectively the order of arrangement of the three known genes of the third linkage group.

[1] The cultures on which this paper is based have been supported in part by grants from the Elizabeth Thompson Science Fund, the American Association for the Advancement of Science, and the Bache Fund of the NATIONAL ACADEMY OF SCIENCES.

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SOMATIC SEGREGATION PRODUCED BY X-RAYS IN DROSOPHILA MELANOGASTER

BY J. T. PATTERSON

UNIVERSITY OF TEXAS, DEPARTMENT OF ZOÖLOGY

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The term *somatic segregation* has been applied to cases in which genetically diverse tissues are found in the soma of the same individual, irrespective of the causes underlying the production of such tissues. But as Babcock and Clausen have pointed out, it would be better to restrict its use to such cases as involve an "actual segregation of the chromosomes among somatic cells." (*Genetics in Relation to Agriculture*, 2nd Ed., p. 363.) If used in the latter sense, then it can be stated that somatic segregation can be induced by x-rays in Drosophila.

The evidence for this conclusion was obtained in a series of x-ray experiments in which egg and larval stages were treated. The eggs and larvae came from a cross between yellow-white females and eosin singed males. The F_1 females developing from these eggs will have gray body color, light eosin eye color, and the bristles and hairs will not be singed. As I have elsewhere pointed out (*J. Heredity*, **20**, p. 261), some of these females will show variant areas, due, for the most part, to a break of one of the X-chromosomes, followed by the elimination of the broken-off piece. From the nature of the cross used, two kinds of variant areas on the body of the fly may be expected, as follows: (a) a yellow non-singed area, if the left end of the gray eosin singed chromosome is eliminated; (b) a gray-singed area, if the left end of the yellow-white chromosome is lost. Also there will be variant area produced on the eyes, but they will not be considered here.

In all, eighteen different cultures of eggs and young larvae were treated, and from these there developed 1137 F_1 females. One hundred and twenty-two females showed variant areas of one or the other kind, or else a combination of the two. The records show that ninety-six females had a single gray-singed area each, and that eighteen had a single yellow area each. One had two distinct and separate gray-singed areas, and seven had, in each case, a combination of a gray-singed area adjacent to a yellow non-singed area. There was no fly with two separate yellow areas, and none was found with a yellow area separated from a gray-singed area.

It is the seven cases that constitute the evidence for somatic segregation. The best way to identify the variant areas is by the character of the hairs and bristles, and the following descriptions of the seven cases are based upon these characteristics. Case 1—The post-alars were yellow non-singed, the supra-alars, notopleurals and twenty-two associated hairs were gray singed, all on the right side of thorax. Case 2—The posterior dorsocentral and eleven hairs, extending to the posterior margin of the thorax, were gray singed, the left scutellars were yellow non-singed, all on the left side. Case 3—The post-alars and two hairs were gray singed, the supra-alars and ten hairs were yellow, all on the left side. Case 4—The dorsocentrals, supra-alars and thirty hairs were yellow non-singed, the presutural and nine hairs were gray singed, all on the right side. Case 5—One yellow non-singed and three adjacent gray-singed bristles were present on the posterior margin of the 4th abdominal segment, left side. Case 6—The upper humeral was gray singed and a single adjacent hair was yellow non-singed, both on the right side. Case 7—Two bristles and two hairs were yellow non-singed, and one adjacent bristle was gray singed, on the posterior margin of the 3rd abdominal segment, left side.

The fact that in every case in which a fly showed a yellow non-singed and a gray-singed area the two areas were contiguous, at once suggests that they had arisen from sister cells. The best explanation is that at a division of an early embryonic or larval cell, one daughter cell received both halves of the yellow-white chromosome and the other received both halves of the eosin singed chromosome. The two groups of descendant cells would then give rise to the two adjacent areas. Another explanation would be that two adjacent embryonic cells were affected by the X-rays, resulting in one of the cells losing a part (or all) of the yellow-white chromosome, and the other losing a part (or all) of the eosin singed chromosome.

The latter possible explanation becomes highly improbable if we consider all of the data on variant areas found among the 1137 females. There were only eighteen flies each with a single yellow non-singed area as against ninety-six each with a single gray-singed area. Singed areas are found much more frequently than yellow areas—at a ratio of more

than five to one. This may be partly due to the singed areas being more conspicuous but, as I have elsewhere pointed out, the disproportion between the two types of areas probably indicates also that under the influence of X-rays the yellow-white chromosome breaks more easily than the eosin singed chromosome. In the case of the flies with two areas, seven out of eight had combined areas, and this gives nine gray-singed areas to seven yellow non-singed areas, or a ratio of almost one to one, instead of five to one. Furthermore, if the X-rays happened to affect two distinct cells, in the same embryo, it is improbable that in seven out of eight cases they would lie adjacent or close to each other. For these reasons the second suggestion seems untenable.

It is impossible to study cytologically such cases as described above, so that one cannot state definitely the nature of the mechanism which produces the segregation. It almost certainly involves some form of synapsis ("pseudosynapsis") between the two X-chromosomes in the somatic cells, so that when the division occurs the two divided chromosomes of the tetrad-like figure pass to opposite poles of the spindle.

THE RADIAL VELOCITIES OF GLOBULAR CLUSTERS

BY P. TEN BRUGGENCATE

THE MOUNT WILSON OBSERVATORY, CARNEGIE INSTITUTION OF WASHINGTON

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The following discussion of the radial velocities of globular clusters originated in conversations which I had with Dr. Zwicky of the California Institute of Technology during my stay at the Mount Wilson Observatory. Since it depends essentially on the validity of Strömberg's solution for the motion of the sun relative to the system of the globular clusters,[1] it will be useful to consider this solution, the more so because v. d. Pahlen and Freundlich[2] have objected to it and have concluded that the motion of the sun is probably in a direction almost at right angles to that found by Strömberg.

1. It seems strange, according to v. d. Pahlen and Freundlich, that the radial velocities of clusters, which are nearly all situated in one direction, give for the sun a large relative motion which is almost perpendicular to the direction of the clusters. They therefore assume that such a solution as Strömberg's cannot be real, and try to solve the problem by taking the simple arithmetical mean of the observed radial velocities. Since this mean value is negative, they conclude that the sun is moving toward the center of the group of clusters, and thus almost perpendicu-

larly to the direction found by Strömberg. But I think their treatment is erroneous, for the following reasons:

Let ξ, η, ζ be the components of the velocity of the sun relative to a group of celestial objects, and V_s the peculiar radial velocity of a celestial object belonging to the group. Then the observed radial velocity of the object (V) may be written in the form

$$V = a\xi + b\eta + c\zeta + V_s . \qquad (1)$$

The quantities ξ, η, and ζ can always be determined by a least-squares solution, provided the values of V_s may be treated as accidental errors; that is, provided we may assume that the members of the group have no common stream motion. On the other hand, if we form the mean value of the expressions (1), we have

$$\overline{V} = \bar{a}\xi + \bar{b}\eta + \bar{c}\zeta + \overline{V}_s . \qquad (2)$$

In the case of an unequal distribution of the celestial objects over the sky the mean values $\bar{a}$, $\bar{b}$, $\bar{c}$ will not in general vanish. If $\overline{V}_s \neq 0$ we have the same difficulty as with Strömberg's solution; if $\overline{V}_s = 0$ it still remains impossible to decide whether the observed negative mean value of $\overline{V}$ is caused by a motion of the sun in the direction of ξ, η, or ζ. It should be stated that v. d. Pahlen and Freundlich noticed a dependence of the distribution of positive and negative radial velocities of the clusters on galactic latitude and longitude, so that $\overline{V}_s$ may perhaps be different from zero. In this case no further investigation can be made by either method, whereas as soon as we assume (as we shall assume) that the single members of the group of globular clusters do not possess any common stream motion ($\overline{V}_s = 0$), Strömberg's solution is the only possible one.

2. From the work of Hubble[3] and Humason[4] it is known that the radial velocities of spiral nebulae show a linear correlation with the distances. The red-shift of the spectral lines increases by an amount corresponding to 500 km./sec. for an increase of 10^6 parsecs in distance. Evidently the observed radial velocities of spirals cannot be interpreted as real motions without making very artificial assumptions as to the dynamical properties of the universe. Difficulties also arise if the increase in red-shift with increasing distance is to be explained geometrically on the assumption that our world has the properties of de Sitter's solution of Einstein's equations. This possibility has been discussed by Silberstein and recently by Tolman.[5] Another suggestion, namely, that the light of very distant objects is reddened in some physical way on its long journey through space, has recently been made by Zwicky.[6] Such a physical effect will depend naturally on the density of matter in interstellar space and on the distance of the celestial object.

If the observed radial velocities of the spirals can be explained by some

physical process depending on the total amount of matter between us and the spirals, then a similar red-shift should be found for globular clusters, since the much larger density inside of the galactic system compensates the much smaller distances of the globular clusters.

Since the effect depends on two variables, one the mean density between us and the celestial object, the other the distance, the empirical treatment of the problem is somewhat difficult. Fortunately, however, in the case of spiral nebulae the variations in the mean density of matter in interstellar space from nebula to nebula are small in comparison with the variations in distance. In this case, therefore, the empirical relation between *red-shift* and *distance* can be found (Hubble's correlation) by assuming a constant density of matter in interstellar space. On the other hand, for the globular clusters whose radial velocities are known the

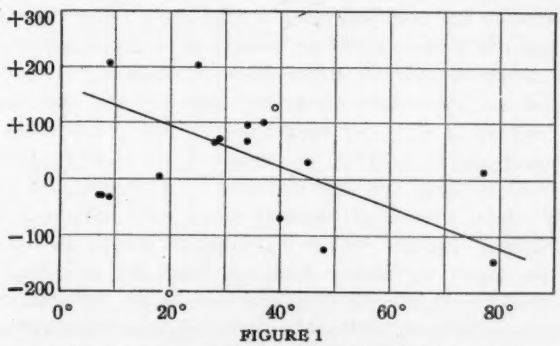

FIGURE 1

Radial velocities corrected for solar motion (ordinates) plotted against galactic latitudes.

variation in distance from cluster to cluster is small in comparison with the variation of density inside the galactic system. In this case, therefore, the empirical relation between *red-shift* and *density* can be found by assuming a constant distance for the clusters in question.

A first rough indication for the density of matter between us and a cluster will be the galactic latitude of the cluster. In figure 1 these latitudes are plotted against the radial velocities of the clusters, corrected for solar motion as found by Strömberg. The residuals resulting from Strömberg's solution show a slight dependence on galactic latitude, the red-shift being larger for clusters in small latitudes.[7] Thus, the correlation is of the kind predicted by Zwicky's explanation; nevertheless, a dynamical interpretation of this correlation cannot be ruled out *a priori*. If for instance the clusters are all falling freely toward the center of the larger galactic system in the gravitational field of its massive center, such a

correlation will appear, since all the clusters whose radial velocities have been measured accurately are probably nearer than the distant center of the galactic system.

We thus have three possibilities:

(a) We interpret the correlation for the clusters on the same physical basis as that for the spirals.

(b) We interpret the correlation for the clusters dynamically. Then, however, we are left without explanation of the red-shifts shown by the spirals, because in this case neither Zwicky's hypothesis nor any other depending on the total amount of matter between us and the celestial object is valid.

(c) We interpret the correlation as a superposition of a dynamical effect and that proposed by Zwicky. Zwicky's explanation then remains valid for the spirals.

3. We discuss the first possibility a little more fully: For numerical investigations it will be much better to take as an indication of the mean density of matter between us and a cluster the number of stars inside a cone of a certain solid angle whose axis joins the sun and the cluster. Since the number of stars per square degree does not decrease *linearly* with increasing galactic latitude, we cannot expect to find a *linear* correlation between red-shift and star numbers for a cone of any arbitrarily chosen solid angle. We must therefore find a solid angle which gives a linear correlation. For our purpose it is sufficient to take the star numbers (N) from the mean distribution table of Seares and van Rhijn.[8] Star counts down to the eighteenth magnitude were used, and solutions based on the observed radial velocities (V) of the globular clusters were made in the form

$$V = a\xi + b\eta + c\zeta + \frac{N}{10^7}\, v + \mathrm{K}\,. \qquad (3)$$

ξ, η, ζ being the components of the relative velocity of the sun, v the red-shift, which is to be found, K a constant term, and N the number of stars inside the cone of given solid angle. Cones with angles of 1°, 30°, and 60° were used. Since the tabulated star numbers refer to one square degree, the numbers for 30° and 60° cones had to be derived by numerical integration.

Let β_0, λ_0 be the galactic coördinates of a cluster, 2α the solid angle of the cone, and $f(\beta)$ the mean distribution of stars with galactic latitude. Then

$$N = \int_{\beta_0 - \alpha}^{\beta_0 + \alpha} \int_{\lambda_1(\alpha, \beta_0, \lambda_0)}^{\lambda_2(\alpha, \beta_0, \lambda_0)} f(\beta) \cos\beta \, d\beta \, d\lambda$$

or, since

$$\cos\alpha = \sin\beta \sin\beta_0 + \cos\beta \cos\beta_0 \cos(\lambda - \lambda_0),$$

we find $$N = 2\int_{\beta_0-\alpha}^{\beta_0+\alpha} f(\beta)\cos\beta \arccos\frac{\cos\alpha - \sin\beta\sin\beta_0}{\cos\beta\cos\beta_0}\,d\beta. \qquad (4)$$

The integration was performed numerically. A definite value of v, and hence a *linear* correlation between red-shift and star numbers, was found only for solid angles of 60°. The correlation also holds for larger angles, but for those much smaller than 60°, the star numbers increase too fast with decreasing latitude.

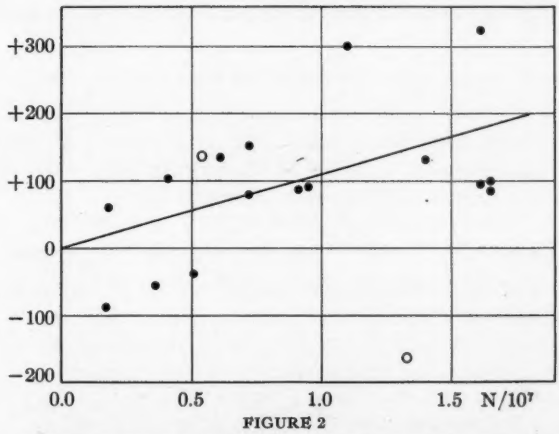

FIGURE 2

Values of $V-(K+a\xi+b\eta+c\zeta)$ plotted against star numbers (abscissae). Circles indicate uncertain values. The straight line represents the least-squares solution.

The solution for solid angles of 60° was also made without a K term. The results, derived from 16 clusters (two clusters with uncertain velocities have been excluded) are:

With K term	*Without K term*
$\xi = -41 \pm 58$ km./sec.	$\xi = -43 \pm 56$ km./sec.
$\eta = +131 \pm 63$	$\eta = +129 \pm 61$
$\zeta = -171 \pm 102$	$\zeta = -197 \pm 91$
$v = +109 \pm 79$	$v = +73 \pm 50$
$K = -37 \pm 61$	$K = 0$

The two solutions are practically identical, and give a mean red-shift of about

$$v = +91 \pm 65 \text{ km./sec.} \qquad (5)$$

The negative K-term in the first solution cannot be considered as real.

The values of $V - (K + a\xi + b\eta + c\zeta)$ and of $V - (a\xi + b\eta + c\zeta)$ are plotted against the star numbers N in figures 2 and 3. In view of the uncertainty in the observed radial velocities and the roughness with which the number of stars counted represents the mean density of matter in space, the solutions satisfy the observations very well. The values for the two excluded clusters with uncertain radial velocities are plotted in the figures as open circles.

4. In interpreting the correlation (3) for the clusters on the same physical basis as for the spirals, it must be shown that the observed red-shift (5) is of the order of magnitude predicted by Zwicky's formula:

$$\frac{\Delta\nu}{\nu} = \alpha\,\Gamma\,\rho\,\frac{l^2}{c^2}, \tag{6}$$

where Γ is the gravitational constant, ρ the mean density of matter inside the cone, l the distance traveled by the light quanta, c the velocity of light, and α a constant of the order of unity.

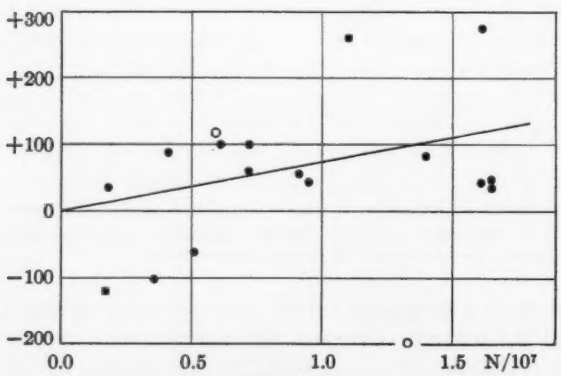

FIGURE 3

Results similar to those of Fig. 2 with the K term omitted from the solution.

From the least-squares solutions we have found $v = +0.9 \times 10^7$ cm./sec. for $N = 1.0 \times 10^7$. The counted number N is approximately proportional to the *total* number of stars inside the cone (N_t), and v is proportional to the real red-shift v_t. Hence

$$Nv = N_t v_t. \tag{7}$$

Since

$$\frac{\Delta\nu}{\nu} = \frac{v_t}{c} \text{ and } \rho = 3.8 \times 10^{33}\,\frac{N_t}{l^3} \text{ gr./cm.}^3,$$

which assumes that the mean mass of stars within the cone of 60° solid angle equals the mass of the sun (2×10^{33} gr.), Zwicky's formula (6) gives

$$v_t = \alpha \Gamma (3.8 \times 10^{33}) \frac{N_t}{lc} \text{ cm./sec.} \qquad (8)$$

For $N = 1.0 \times 10^7$ we have

$$N_t v_t = 3.8 \times 10^{33} \alpha \Gamma \frac{N_t^2}{lc} = 0.9 \times 10^{14}.$$

With

$l = 16{,}000$ parsecs $= 5 \times 10^{22}$ cm., $c = 3 \times 10^{10}$ cm./sec., $\Gamma = 6.7 \times 10^{-8}$,

we find

$$\alpha N_t^2 = 0.6 \times 10^{21}.$$

Since α is of the order of unity, this gives

$$N_t = \beta \times 10^{10}.$$

Whence

$$N_t = \beta \times 10^3 \times N, \qquad (9)$$

β being a constant of the order of unity.

This means that the total number of stars required to bring the observed value of the red-shift into agreement with Zwicky's theoretical result is 1000 times the number brighter than the eighteenth magnitude. The direct evidence bearing on this point is rather slender and depends on an uncertain extrapolation of the behavior of the stars which can be observed. The mean distribution tables of Seares and van Rhijn[8] indicate for stars in the galactic plane a value of about 500 for the above ratio. This is a mean result, averaged over all longitudes, and is not directly applicable because of the eccentric position of the sun within the galactic system. Since most of the clusters are in the general direction of the center of the stellar system, the ratio of 500 must be increased by an unknown factor, perhaps as large as two or three. Non-luminous and obscured matter will also contribute something. On the other hand, the fact that the latitudes are not zero will diminish the ratio derived from the mean distribution table. A balancing of these opposing tendencies leads to the conclusion that the theoretical ratio of 1000 to 1 found above is of the right general order of magnitude to reconcile the observed red-shift with Zwicky's prediction.

5. Our results may be summarized as follows:

(a) There exists a linear correlation between the displacements of spectral lines for globular clusters, corrected for solar motion, and galactic latitude, similar to the correlation between line displacement and distance which holds for the spirals.

(b) The correlation may express a dynamical property of the larger galactic system.

(c) The correlation may express a physical relation between red-shift and total amount of matter between us and the clusters. Proceeding on this assumption, we find a linear correlation between red-shift and star numbers in cones with solid angles of 60° or larger. The amount of the red-shift seems to be of the general order required by Zwicky's formula.

I wish to express my best thanks to Dr. Hubble and Dr. Strömberg, and especially to Profesor Seares and Dr. Zwicky, for many suggestions and much valuable information relating to the subject. Also my best thanks are due to Dr. Adams for permission to publish my results in this way.

[1] Gustaf Strömberg, *Mt. Wilson Contr.*, No. 292; *Astrophys. J.*, Chicago, Ill., **61**, 1925 (353–362).

[2] E. von der Pahlen and E. Freundlich, *Pub. Astrophys. Obs.*, Potsdam, **26** (No. 86), 1928.

[3] Edwin Hubble, *Mt. Wilson Comm.*, No. 105; these PROCEEDINGS, **15**, 1929 (168–173).

[4] M. L. Humason, *Mt. Wilson Comm.*, No. 104; these PROCEEDINGS, **15**, 1929 (167–168).

[5] R. C. Tolman, *Astrophys. J.*, Chicago, Ill., **15**, 1929 (297–304).

[6] F. Zwicky, these PROCEEDINGS, **15**, 1929 (773–779).

[7] If the observed radial velocities are plotted directly against galactic latitude, the correlation is much worse.

[8] F. H. Seares, P. J. van Rhijn, et al., *Mt. Wilson Contr.*, No. 301, Table XVII; *Astrophys. J.*, Chicago, Ill., **62**, 1925 (320–374).

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*SOLID PROTEIN HYDROCHLORIDES**

BY WILDER D. BANCROFT AND C. E. BARNETT

BAKER LABORATORY OF CHEMISTRY, CORNELL UNIVERSITY

Communicated December 26, 1929

If one adds hydrogen chloride gas at a constant temperature to a solid and if one plots the amount of hydrogen chloride taken up by unit mass of the solid as abscissas against the equilibrium pressures as ordinates, it is an easy matter to tell whether hydrogen chloride forms a definite chemical compound with the solid according to stoichiometric proportions or whether the hydrogen chloride is adsorbed by the solid. If a definite chemical compound is formed, we know from the phase rule that the equilibrium pressure must remain constant so long as there are two solid phases present, the pure solid and a compound, or two compounds. A flat in the isotherm as plotted means the existence of a second solid phase. If there is adsorption and no definite formation of a compound according to stoichiometric proportions, the isotherm will be a smooth curve bending away from the axis of abscissas at higher concentrations.

Hexamethylene tetramine has the composition $(CH_2)_6N_4$ and each nitrogen is able to add on one molecule of hydrogen chloride, as can be seen in figure 1, where four flats are shown. In this particular case the four dissociation pressures differ, each from the next one, by amounts which lie outside the limits of experimental error and consequently it is easy to recognize the four flats. It would have been possible for these dissociation pressures to have differed by very little, in which case we should apparently have had a single flat having the same length as the four flats. This would not have obscured the fact that hydrogen chloride forms at least one definite chemical compound; but one could not have told that there were four compounds without having more information than the diagram gives. If, however, one knows the number of nitrogen atoms in the molecule, one can tell how many definite chemical compounds are formed. We know this for hexamethylene tetramine; but we do not know it for the proteins.

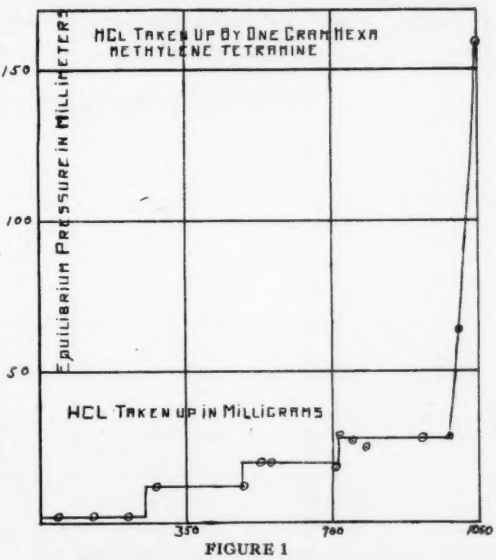


FIGURE 1

In figure 2 is given the isotherm for hydrogen chloride and a special sample of activated charcoal, the temperature in all these experiments being room temperature because there was nothing in the work which made the use of a thermostat either necessary or desirable. No one would expect hydrogen chloride to form a definite compound with carbon and the curve is an obvious adsorption isotherm. As is so often the case in adsorption isotherms, there is a distinct hysteresis, the curve for addition of hydrogen chloride (circles) not coinciding with the curve for the withdrawal of hydrogen chloride (triangles).

Through the courtesy of Professor R. A. Gortner of the University of Minnesota we received samples of five highly purified proteins which he had used in his own work. They were arachin, edestin, fibrin, gliadin, and zein. They were used as received and we are greatly indebted to Professor Gortner. The casein was prepared in the Cornell laboratory.

In figure 3 is shown the isotherm for gliadin. There are three flats, showing that hydrogen chloride forms at least three definite compounds with gliadin. The lengths of these flats stand quite closely in the ratio 2 : 3 : 4. Since hydrogen chloride cannot combine stoichiometrically with a fraction of an atom of nitrogen, the nitrogen atoms which combine stoichiometrically with hydrogen chloride along the three flats must therefore stand in the ratio 2 : 3 : 4. There must, therefore, be some multiple of nine nitrogen atoms (2 + 3 + 4) in the gliadin molecule which combine stoichiometrically with hydrogen chloride. The equivalent weight referred to one active nitrogen works out to about 250 in round numbers.

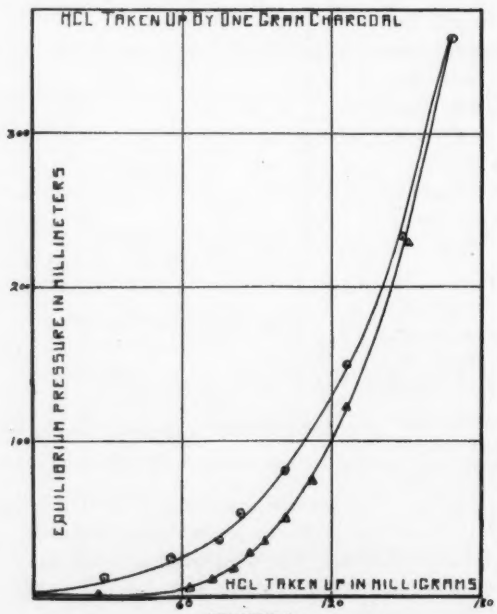


FIGURE 2

Since there must be some multiple of nine active nitrogens in gliadin, its molecular weight must be some multiple of 2250 in round numbers.

As the other extreme we may take zein, for which the isotherm is shown in figure 4. It is a typical adsorption curve with no signs of a definite chemical compound, although the zein takes up fully half as much hydrogen chloride as the gliadin does.

Arachin, casein, edestin and fibrin form definite solid compounds with hydrogen chloride.

The data, taken from the smoothed curves are shown in table 1, where

the concentrations are milligrams of hydrogen chloride taken up by one gram of protein at room temperature and the pressures are millimeters of mercury.

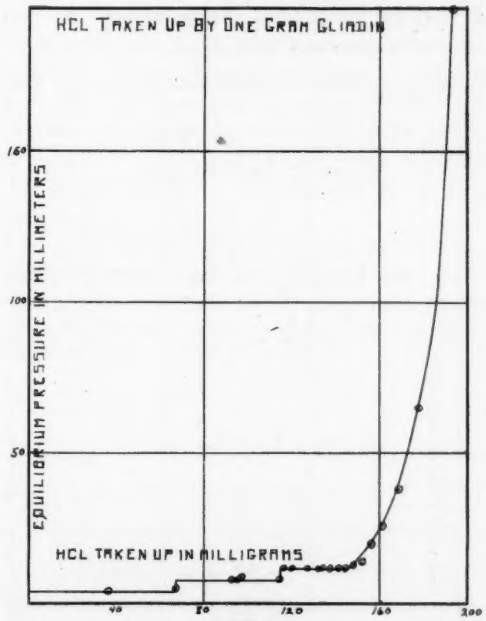


FIGURE 3

With edestin there is at first an adsorption followed by a flat. This introduces an uncertainty into the determination of the equivalent weight,

TABLE 1

ISOTHERMS FOR SIX PROTEINS

A = arachin; C = casein; E = edestin; F = fibrin; G = gliadin; Z = zein.

MG./G. CONC.	MM. A	MM. C	MM. E	MM. F	MM. G	MM. Z
10	6.5	4.5	6.0	6.0	4.0	1.0
20	6.5	4.5	9.0	6.0	4.0	1.5
40	6.5	4.5	9.0	6.0	4.0	11.0
80	6.5	4.5	9.0	6.0	8.0	105.0
100	6.5	23.0	9.0	8.0	8.0	215.0
120	6.5	82.5	13.0	15.0	12.0	...
140	13.5	..	30.0	29.0	12.0	...
160	34.0	..	75.0	80.0	25.0	...

because we do not now know whether the last compound formed between edestin and hydrogen chloride does or does not adsorb as much hydrogen

chloride as the edestin did. Depending on which assumption we make, the equivalent weight of edestin comes out 330 or 405, with the 330 the more probable value. The equivalent weight for arachin comes out 280, for casein 435 and for fibrin 365. These values are uniformly lower than those obtained in aqueous solution.

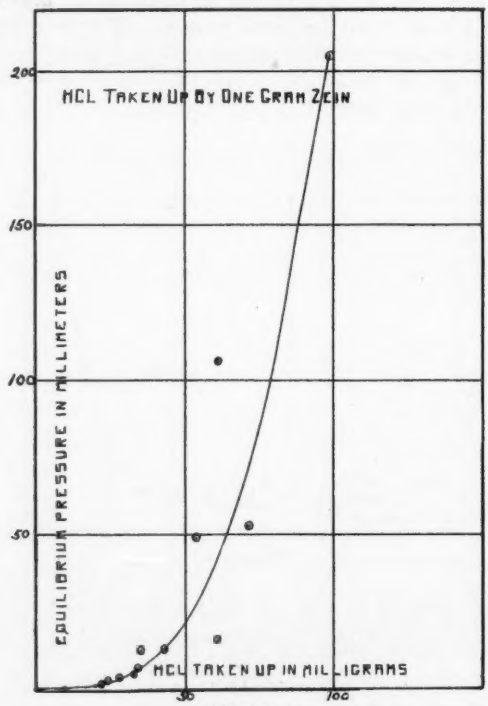


FIGURE 4

The general results of this paper may be summed up as follows:

1. The pressure-concentration isotherms for hydrogen chloride and a solid show clearly whether a definite chemical compound is or is not formed.

2. Hexamethylene tetramine reacts with four mols of hydrogen chloride to give a curve with four flats in it.

3. Charcoal and hydrogen chloride give the smooth curve typical of adsorption.

4. Zein and hydrogen chloride give an isotherm like that for charcoal and hydrogen chloride, no definite chemical compound being formed.

5. Arachin, casein, edestin, fibrin, and gliadin each forms one or more

definite chemical compounds with hydrogen chloride. The gliadin isotherm shows three flats, corresponding to at least nine definite compounds unless one postulates a polymerization of gliadin along the second flat.

6. The equivalent weights come out 280 for arachin, 435 for casein, 330–405 for edestin, 365 for fibrin and 250 for gliadin. Since there are apparently at least nine active nitrogens per molecule of gliadin, the molecular weight of gliadin must be some multiple of 2250. The diagrams give no clue at present as to the molecular weights of the other proteins.

7. It is not safe at present to reason from the behavior of the solid proteins with hydrogen chloride to the behavior of the same proteins in presence of aqueous hydrochloric acid.

8. We are indebted to Professor R. A. Gortner of the University of Minnesota for the specimens of arachin, edestin, fibrin, gliadin, and zein.

* This work is part of the programme now being carried out at Cornell University under a grant from the Heckscher Foundation for the Advancement of Research established by August Heckscher at Cornell University.

THE STRUCTURE OF THE MICAS AND RELATED MINERALS

BY LINUS PAULING

GATES CHEMICAL LABORATORY, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated January 16, 1930

With the aid of the general principles[1] governing the structures of complex ionic crystals I have formulated a structure for talc, pyrophillite, the micas, and the brittle micas which is substantiated by the x-ray examination of the minerals, explains their remarkable physical properties, and leads to a general chemical formula unifying the widely varying analyses reported for different specimens. This structure is described in the following paragraphs.

The monoclinic (pseudohexagonal) unit of structure of muscovite, $KSi_3Al_3O_{10}(OH,F)_2$, has been reported by Mauguin[2] to have $a = 5.17$ Å, $b = 8.94$ Å, $c = 20.01$ Å, and $\beta = 96°$. This unit contains 4 molecules of the above composition. We have verified this with oscillation and Laue photographs of fuchsite, a variety of muscovite; the unit found having $a = 5.19$ Å, $b = 8.99$ Å, $c = 20.14$ Å, and $\beta = 96°$.

The dimensions of the unit in the basal plane closely approximate those for the similarly pseudohexagonal crystal hydrargillite, $Al(OH)_3$, as well as of the hexagonal layers in two forms of silica, β-tridymite and β-cristobalite. The monoclinic (pseudohexagonal) unit of structure of hydrargillite[3] has $a = 8.70$ Å, $b = 5.09$ Å, $c = 9.76$ Å, and $\beta = 85°29'$, and contains $8Al(OH)_3$. The crystal is composed of layers of octahedra,

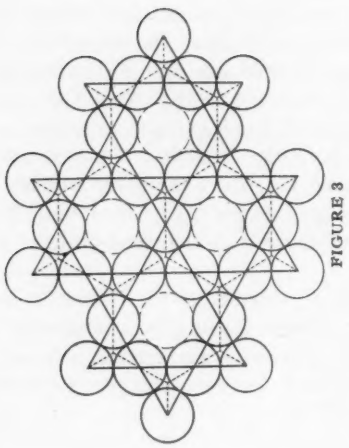


FIGURE 3

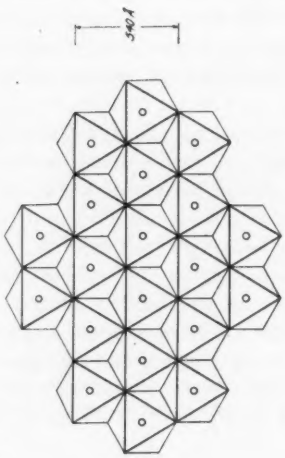


FIGURE 4

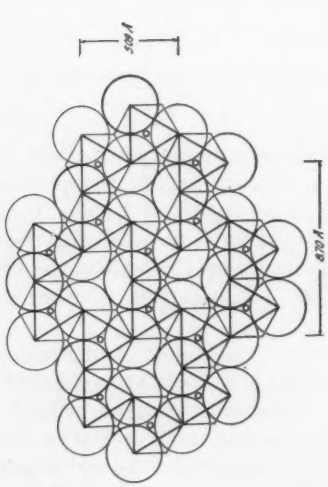


FIGURE 1

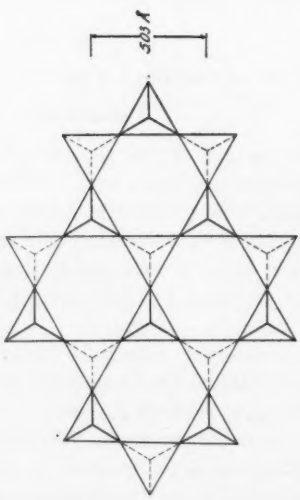


FIGURE 2

Fig. 1. A hydrargillite layer of octahedra. The light circles indicate oxygen ions, the heavier ones hydroxyl or fluorine ions in mica.

Fig. 2. A tetrahedral layer from β-cristobalite or β-tridymite. A silicon ion is located at the center of each tetrahedron, and an oxygen ion at each corner.

Fig. 3. A tetrahedral layer in which all the tetrahedra point in the same direction.

Fig. 4. A complete layer of octahedra (brucite layer).

each consisting of 6 OH^- ions grouped about an Al^{+3} ion, with each octahedron sharing three edges as shown in figure 1. The electrostatic valence rule is satisfied; each OH^- ion is held by two bonds of strength $s = 1/2$. These layers are superimposed, sharing no octahedral elements with one another.

In β-tridymite and β-cristobalite[4] there are present the hexagonal layers of silicon tetrahedra shown in figure 2. Their dimensions, $a = 5.03$ Å and $b = 8.71$ Å, agree closely with those of the hydrargillite layers and of mica. Another type of tetrahedral layer with the same dimensions can be obtained by pointing all the tetrahedra in the same direction (figure 3). The oxygen ions forming the bases of the tetrahedra have their valences satisfied: the strength s of a silicon bond is 1, and each $O^=$ is held by two such bonds, giving $\Sigma s = 2$. But the oxygen ions at the unshared tetrahedral corners have $\Sigma s = 1$ only; it is hence necessary that they be held by further bonds. Now the relative positions of these tetrahedron corners are such that the tetrahedral layer can be imposed on the hydrargillite layer with the tetrahedron corners coincident with two-thirds of the shared octahedron corners; namely, those indicated in figure 1 by the large light circles. These positions are occupied by oxygen ions, which have $\Sigma s = 1 + 1/2 + 1/2 = 2$. The remaining positions, indicated by heavy circles, have $\Sigma s = 1/2 + 1/2 = 1$, and are occupied by hydroxide or fluoride ions. A similar tetrahedral layer is attached to the other side of the hydrargillite layer.

The resultant layer, with a total thickness of about 10 Å, is electrically neutral. A crystal built up by the superposition of such layers would have the composition $Si_4Al_2O_{10}(OH)_2$. It is very probable that the mineral pyrophillite, with this composition, has this structure.

Brucite, $Mg(OH)_2$, is built of complete octahedral layers such as that shown in figure 4, with dimensions not much greater than those of a hydrargillite layer, and with the octahedron corners in the same positions. A neutral layer containing tetrahedra and octahedra closely similar to that described above can be made with this complete octahedral layer in place of the hydrargillite layer; the electrostatic valence rule will be again satisfied, $O^=$ having $\Sigma s = 1 + 1/3 + 1/3 + 1/3 = 2$ and OH^- having $\Sigma s = 1/3 + 1/3 + 1/3 = 1$. These layers probably occur in talc, $Si_4Mg_3O_{10}(OH)_2$, which is reported to be monoclinic, pseudohexagonal, and morphologically similar to mica.

By replacing one-fourth of the silicon ions in a pyrophillite layer by aluminum ions, which can have the coördination number 4 as well as 6, the layers become negatively charged. To regain neutrality further positive ions must be introduced, such as K^+ or Na^+. There is room for these ions between the layers, in the pockets formed by six oxygen ions on the top of one layer and six on the bottom of the layer above. The

composition of such a crystal is $K \cdot Si_3Al \cdot Al_2O_{10}(OH, F)_2$, which is the formula generally assigned to muscovite. A similar treatment of talc gives $KSi_3AlMg_3O_{10}(OH, F)_2$, which is biotite.

Pyrophillite, talc, muscovite, and biotite have the following sequence of atom-planes along the pseudo-hexagonal axes:

Pyrophillite	Talc
$6O^=$	$6O^=$
$4\ Si^{+4}$	$4\ Si^{+4}$
$4O^= + 2OH^-$	$4O^= + 2OH^-$
$4Al^{+3}$	$6Mg^{++}$
$4O^= + 2OH^-$	$4O^- + 2OH^-$
$4Si^{+4}$	$4Si^{+4}$
$6O^=$	$6O^=$

Muscovite		Biotite	
$z_2\ 6O^=$		$6O^=$	
$u\ 3Si^{+4} + Al^{+3}$		$3Si^{+4} + Al^{+3}$	
$z_1\ 4O^= + 2(OH^-, F^-)$		$4O^= + 2(OH^-, F^-)$	
Origin $4Al^{+3}$	Charge -2	$6Mg^{++}$	Charge -2
$4O^= + 2(OH^-, F^-)$		$4O^= + 2(OH^-, F^-)$	
$3Si^{+4} + Al^{+3}$		$3Si^{+4} + Al^{+3}$	
$6O^=$		$6O^=$	
$2K^+$	$+2$	$2K^+$	$+2$
$6O^-$	-2	$6O^=$	-2
$3Si^{+4}$, etc.		$3Si^{+4}$, etc.	

The verification of the suggested structures by the comparison of the observed and calculated intensities of reflection of x-rays has been begun. So far the calculations have been carried out for 18 even orders of re-

TABLE 1

ATOMIC A—VALUES

d/n =	10	5	2.5	1.0	0.75	0.60Å
A_O =	30	21	10	2.0	0.8	0.25
A_{Al} =	49	28	17	5.2	3.0	1.7
A_{Si} =	53	30	18	6.0	3.7	2.1
A_K =	72	45	24	7.8	5.0	3.2

flection from (001). The observed intensities given in table 2 were obtained by the visual comparison of four photographs from fuchsite, identical except for varying exposure times of 15, 90, 300, and 960 minutes. The calculated intensities were obtained from the formula

$$I = \text{constant} \cdot A^2,$$

$$\text{with} \quad A = \Sigma_n A_n e^{2\pi i\ (hx_n + ky\ + lz_n)}, \qquad (1)$$

the summation being taken over all the atoms in the unit. Values of the atomic amplitude functions[5] used are given in table 1. Assuming the group of atoms such as $4O^{=} + 2(OH^{-}, F^{-})$ to depend on a single parameter, only three parameters are involved, z_1, u, and z_2. Giving the hydrargillite layer the thickness found for it in hydrargillite, and assuming the tetrahedra to be regular and 2.60 Å on an edge as in silicates in general, the parameter values $z_1 = 0.055$, $z_2 = 0.165$, and $u = 0.137$ were predicted.

The amplitude expression is

$$A = 2A_{Al} + (-1)^{l/2} A_K + (3A_{Si} + A_{Al}) \cos 2\pi lu + 6A_O(\cos 2\pi lz_1 + \cos 2\pi lz_2), \quad (2)$$

with l having even values from 2 to 36. It was found that the values of A^2 were in general agreement with the observed intensities for the predicted parameter values. Slightly better agreement was obtained by reducing u to 0.135. The calculated intensities (the constant in equation 1 being given the arbitrary value 0.015) are included in table 2, and represented graphically together with the observed intensities in figure 5. The general agreement is striking, and lends strong support to the suggested structure.

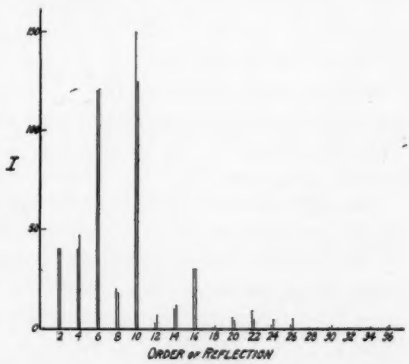


FIGURE 5

Comparison of the observed and calculated intensities of reflection of x-rays from (001) of fuchsite, in the 2nd to 36th order. In each pair of vertical lines that on the left shows the observed intensity, that on the right the calculated intensity.

TABLE 2

OBSERVED AND CALCULATED INTENSITIES OF REFLECTION FROM (001) OF FUCHSITE

ORDER OF REFLECTION	2	4	6	8	10	12	14	16	18
Observed intensity	40	40	120	20	150	3	10	30	0.5
Calculated intensity	40	47	121	18	124	7	12	30	1.4
ORDER OF REFLECTION	20	22	24	26	28	30	32	34	36
Observed intensity	6	9	2	2	0	0.8	0.1	0	1
Calculated intensity	4.3	4.5	4.8	5.2	0.0	1.6	0.2	0.6	1.7

The mica structure provides an interesting variation of the types of close-packing observed for large ions in crystals. The two central layers of $O^{=}$, OH^{-}, and F^{-} ions form close-packed planes with three spheres in a hexagonal unit of edge 5.2 Å, at positions 00, $^1/_3{}^2/_3$, and $^2/_3{}^1/_3$ relative

to axes 120° apart. The outer layers, however, consist of oxygen ions occupying three of the four positions of a close-packed plane with four spheres in the same unit of edge 5.2 Å, at positions such as 00, $0\frac{1}{2}$, and $\frac{1}{2}0$. The insertion of a fourth sphere at $\frac{1}{2}\frac{1}{2}$, indicated by the dotted circles in figure 3, would complete this layer. Such a large variability in the effective dimensions of the large ions introduces some ambiguity in structure determinations involving the deduction of the type of close-packing from the size of the unit of structure.

The structure leads to a general formula for the micas: namely, $KX_nY_4O_{10}(OH,F)_2$, with $2 \leq n \leq 3$, in which X represents cations of coördination number 6 (Al^{+3}, Mg^{+}, Fe^{++}, Fe^{+3}, Mn^{++}, Mn^{+3}, Ti^{+4}, Li^{+}, etc.) and Y cations of coördination number 4 (Si^{+4}, Al^{+3}, etc.). The subscript n can have any value between 2 (hydrargillite layer) and 3 (complete octahedral layer). K^{+} can be partially replaced by Na^{+} and possibly to some extent by Ca^{++}. This formula represents satisfactorily the numerous recently published mica analyses almost without exception.[6] The distribution of the various ions X and Y must be such as to give general agreement with the electrostatic valence rule.

The clintonites or brittle micas have a similar structure, the layers having twice the electrical charge of those in mica, and being held together by calcium ions instead of potassium ions. The correspondingly stronger forces bring the layers closer together, the separation of adjacent layers being 9.5–9.6 Å in place of the value 9.9–10.1 Å for the micas. The general formula $CaX_nY_4O_{10}(OH, F)_2$, with $2 \leq n \leq 3$, holds for these minerals.

The physical properties of talc, pyrophillite, the micas, and the brittle micas are in agreement with the suggested structure. To tear apart one of the pseudohexagonal layers it is necessary to break the strong Si–O, Al–O, etc., bonds; as a consequence these individual layers are tough. But they can be easily separated from one another, giving rise to the pronounced basal cleavage shown by all these minerals. In talc and pyrophillite the layers are electrically neutral, and are held together only by stray electrical forces. These crystals are accordingly very soft, feeling soapy to the touch as do graphite crystals. To separate the layers in mica it is necessary to break the bonds of the univalent potassium ions, so that the micas are not so soft, thin plates being sufficiently elastic to straighten out after being bent. Separation of layers in the brittle micas involves breaking bonds of bivalent calcium ions; these minerals are hence harder, and brittle instead of elastic, but still show perfect basal cleavage. The sequence of hardness is significant; on the Mohs scale it is: talc and pyrophillite, 1–2; the micas, 2–3; the brittle micas, $3\frac{1}{2}$–6.

I wish to acknowledge the assistance of Mr. J. Sherman in the preparation and analysis of the x-ray photographs.

A further account of the x-ray investigation of these minerals and a comparison of the suggested general formulas with the reported chemical analyses will be published in the *Zeitschrift für Kristallographie*.

[1] Linus Pauling, "Sommerfeld Festschrift," S. Hirzel, Leipzig, **1928**, p. 11; *J. Am. Chem. Soc.*, **51**, 1010 (1929).

[2] Ch. Mauguin, *Comp. Rend.*, **185**, 288 (1927).

[3] The determination of the structure of hydrargillite will be reported in a paper to be published in the *Zeits. für Krist.*

[4] For known crystal structures reference may be made to the Strukturbericht of P. P. Ewald and C. Hermann in current numbers of *Zeits. für Krist.*

[5] The atomic amplitude functions take account of the atomic *F*- factor, the temperature factor, the Lorentz factor, and the polarization factor.

[6] Mauguin showed that for various micas the unit of structure uniformly contains the number of atoms $O + F$ given by the above formula.

THE DECOMPOSITION OF NITROGEN PENTOXIDE AT LOW PRESSURES

BY HANS-JOACHIM SCHUMACHER* AND GERHARD SPRENGER

LABORATORY OF PHYSICAL CHEMISTRY, PRINCETON UNIVERSITY, AND TECHNISCHE HOCHSCHULE, DANZIG-LANGFUHR

Communicated December 20, 1929

Introduction.—Nitrogen pentoxide[1] and nitrylchloride[2,3] are the only known inorganic compounds which decompose unimolecularly. Because of their simple molecular structure these two compounds should be well adapted to give direction to the theory of unimolecular reactions. For this reason the kinetics of these reactions have been investigated with the utmost care.

In spite of these studies the velocity of the decomposition of nitrogen pentoxide, at least at low pressures, is still in dispute.[4] For the most part the explanation of the discordant results is to be found in the experimental difficulties involved in working at low pressures with a chemically active substance such as N_2O_5. However, the behavior at low pressures is of special interest.

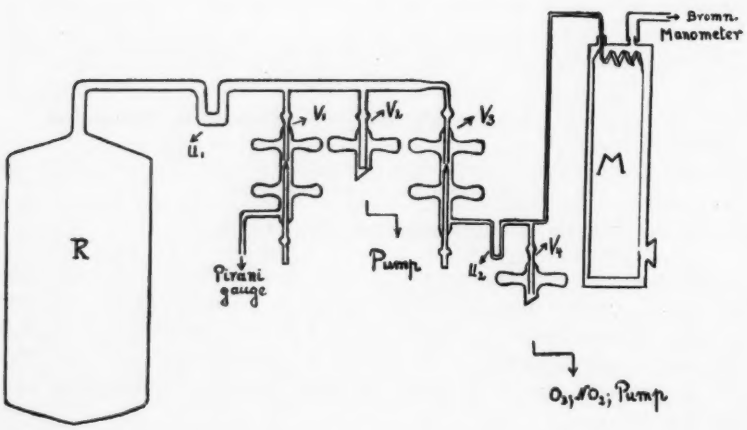
As is well known, the present theory requires that every unimolecular reaction should become bimolecular at a certain pressure. This change is indicated by a falling off in the value of the unimolecular constant. The pressure at which this occurs for different compounds is greatly influenced by the number of degrees of freedom of the molecule.

Recently it has been shown, in the accurate work of Ramsperger, Nordberg and Tolman,[5] that for pressures down to 0.2 mm. Hg the velocity

constant for the decomposition of N_2O_5 retains its normal value. On the basis of the most favorable assumptions Kassel[6] has calculated that at 300°A. the velocity should decrease by 10–15% at a pressure of about 0.05 mm. Hg.

In the work about to be described data will be given for the decomposition of N_2O_5 at pressures less than 0.2 mm. Hg. The discussion of the theoretical significance of these results will be delayed until measurements, now in process of completion, of the specific heat of N_2O_5 have been determined.

Apparatus.—The reaction chamber *R* (Fig. 1) was a cylindrical bulb of soda-glass or clear quartz, having a diameter of about 15 cm. and a capacity of about 5 liters. The U tube U_1, and the glass valves V_1, V_2, V_3 are connected with the reaction chamber by means of 12 mm. glass



tubing. V_1 is connected to a Pirani gauge, V_2 to a mercury diffusion pump and V_3 to a U tube U_2, about 3 mm. diameter, the glass valve V_4 and the quartz manometer *M*. In every case that end of the valve which had a small dead space was attached to the tube leading to the reaction chamber.

The Pirani gauge, immersed in ice water in a Dewar flask, was calibrated with oxygen, the pressure of which was measured by a McLeod gauge. The resistance of the platinum wire in the Pirani gauge was determined by means of a sensitive potentiometer.

The quartz manometer *M* was used as a zero instrument with an accuracy of 0.02 mm. Hg; the real pressure being read on the bromnaphthalene manometer held at 22°C. in a water thermostat.

The glass valves were constructed according to the same principle as Bodenstein's membrane valves. The metal membrane was replaced by two glass spheres of a diameter about 5–6 cm. and with walls of the proper thickness, the ends of the spheres being collapsed until they were about 0.5 cm. apart. The needle consisted of a sealed glass capillary and the seat of thick walled glass tubing, the two being carefully polished with rouge to make a ground glass joint. The glass valve was mounted in a metal case, in such a manner that the glass rings could be drawn apart or pressed together by screwing, thus opening and closing the valve. It was found that the valves were vacuum tight.

A water thermostat or a furnace for baking out the vessel could be raised into position to fit around the reaction chamber.

Experimental Procedure.—Before every experiment the reaction vessel was simultaneously baked out and pumped for about 20 hours; for the glass and quartz vessels the baking out temperatures were, respectively, about 350° and 500°C. Only small quantities of N_2O_5 and its decomposition products entered the reaction chamber, all foreign gases being excluded from the time of the first experiment until the work was complete.

The N_2O_5 to be decomposed in any one experiment was allowed to expand from a small vessel ($V_3 + U_2$) under a known pressure into the reaction chamber. Since the volumes of the two vessels were known, the initial pressure of N_2O_5 could be calculated. The small vessel consisted for the most part of the dead space of the glass valve V_3. The ratio (about 150) of the volumes was accurately determined by calibration with oxygen. The tube U_2 contained a large amount of N_2O_5 frozen in liquid air. Before an experiment the N_2O_5 was evaporated, the pressure was read, a small quantity of the gas was admitted into R thereby causing a pressure decrease of a few mm. in the small vessel. The N_2O_5 was made from NO_2 and O_3 in the small vessel itself.[7]

The decomposition of N_2O_5 was followed by measuring with the Pirani gauge the pressure of oxygen formed after having frozen out the N_2O_5 and NO_2 in U, with liquid air. In using this method it is assumed that oxygen is not appreciably occluded by the frozen oxides of nitrogen.[8] That this assumption is correct under the conditions of the experiments was shown by admitting into R known pressures of O_2 and NO_2 equal in amount to the quantities present during a decomposition, freezing out the NO_2 and reading the oxygen pressure. Within the limits of the experimental error the observed oxygen pressure always equaled the theoretical value.

In order to obtain the purity of the N_2O_5 and also the pressure at time $t = 0$, the oxygen pressure was observed before and immediately after the admission of N_2O_5. The final pressure of the oxygen was found by

heating the reaction vessel at 60°C. for about 40 minutes and subsequently cooling to the temperature of the experiment.

The glass valve V_1 was never open, except to read the oxygen pressure and then only after U_1 had been cooled in liquid air for at least 10 minutes. The Pirani gauge was calibrated twice during the course of the experiments, no appreciable deviations from the first calibration curve being found.

Experimental Results.—In the following tables $P_{\text{theor.}}$ is the N_2O_5 pressure calculated from the quartz manometer reading, $P_{\text{exp.}}$ is the N_2O_5 pressure obtained from the oxygen pressure at the end of the reaction, P_{O_2} is the oxygen pressure, all pressures being given in mm. Hg at the experimental temperature. The velocity constant k was calculated by means of the usual unimolecular equation using $P_{\text{exp.}}$, the time being expressed in minutes.

A total of about 20 experiments was performed at 35°C. The value of k found by Daniels at 35° is: $k_{35°} = 8.0 \times 10^{-3} \frac{1}{\text{min.}}$.

TABLE 1

$P_{\text{theor.}}$ 0.10 MM. HG; $P_{\text{exp.}}$ 0.097 MM. HG	GLASS VESSEL	T: 35°C.
t	$P_{O_2} \times 10^3$	$k \times 10^3$
0	2.57	..
20	10.1	8.5
50	19.0	8.4
90	26.6	7.0
140	33.8	7.3
220	41.9	8.5
	50.1	

k mean $= 7.9 \times 10^{-3} \frac{1}{\text{min.}}$

TABLE 2

$P_{\text{theor.}}$ 0.0638 MM. HG; $P_{\text{exp.}}$ 0.0620 MM. HG	GLASS VESSEL	T: 35°C.
t	$P_{O_2} \times 10^3$	$k \times 10^3$
0	2.69	..
25	10.1	10.9
61	15.6	7.32
111	21.6	7.88
198	28.1	8.65
	33.9	..

k mean $= 7.9 \times 10^{-3} \frac{1}{\text{min.}}$

TABLE 3

$P_{theor.}$ 0.0369 MM. HG — GLASS VESSEL — T: 35°C.
$P_{exp.}$ 0.0364 MM. HG

t	$P_{O_2} \times 10^3$	$k \times 10^3$
0	4.77	..
40	8.86	6.36
92	12.0	4.84
172	16.1	5.84
	23.0	

$$k \text{ mean} = 5.70 \times 10^{-3} \frac{1}{\text{min.}}$$

TABLE 4

$P_{theor.}$ 0.0128 MM. HG — GLASS VESSEL — T: 35°C.
$P_{exp.}$ 0.0108 MM. HG

t	$P_{O_2} \times 10^3$	$k \times 10^3$
0	0.0	..
60	1.58	5.46
140	2.78	4.65
280	4.03	4.54
	5.44	..

$$k \text{ mean} = 4.88 \times 10^{-3} \frac{1}{\text{min.}}$$

TABLE 5

$P_{theor.}$ 0.0322 MM. HG — QUARTZ VESSEL — T: 35°C.
$P_{exp.}$ 0.0282 MM. HG

t	$P_{O_2} \times 10^3$	$k \times 10^3$
0	1.0	..
35	4.47	8.08
80	7.15	5.74
147	9.06	4.59
247	11.96	6.59
	15.11	

$$k \text{ mean} = 5.64 \times 10^{-3} \frac{1}{\text{min.}}$$

TABLE 6

$P_{theor.}$ 0.0151 MM. HG — QUARTZ VESSEL — T: 35°C.
$P_{exp.}$ 0.0090 MM. HG

t	$P_{O_2} \times 10^3$	$k \times 10^3$
0	2.00	..
54	3.00	4.56
64	3.94	4.81
104	4.84	4.17
398	6.17	6.69
	6.51	..

$$k \text{ mean} = 5.00 \times 10^{-3} \frac{1}{\text{min.}} \text{ (with 6.69)}$$
$$4.50 \times 10^{-3} \text{ (without 6.69)}$$

TABLE 7

$P_{theor.}$ 0.0128 MM. HG $P_{exp.}$ 0.00644 MM. HG	QUARTZ VESSEL	T: 35°C.
t	$P_{O_2} \times 10^3$	$k \times 10^3$
0	2.42	..
55	3.82	4.5
69	4.50	3.0
91	5.20	4.5
120	5.50	4.15
	5.64	..

$$k \text{ mean} = 4.05 \times 10^{-3} \frac{1}{\text{min.}} \text{ (with 3.0)}$$

$$4.40 \times 10^{-3} \text{ (without 3.0)}$$

An examination of the above data discloses the fact that the decomposition of N_2O_5 proceeds with the same velocity down to pressures of about 0.06 mm. Hg. At least the velocity constant does not vary more than 10%, which is within the limits of the experimental error at these pressures. However, at lower pressures a decrease in the velocity constant was found. The lowest pressure, at which the decomposition was studied, was 0.004 mm. Hg. The mean value of $k_{35°}$ for this experiment was $4.8 \times 10^{-3} \frac{1}{\text{min.}}$, the accuracy being only 20 per cent at these very low pressures. The mean value of k in this experiment seems to be too high since the average k for an experiment with $P_{N_2O_5} = 0.006$ mm. Hg is only $4.4 \times 10^{-3} \frac{1}{\text{min.}}$. For this reason the slope of the curve obtained by plotting k against p cannot be accurately determined from the present data. There is no difference between the values of k obtained from decompositions of N_2O_5 in glass or quartz vessels.

Furthermore, it was found that $P_{exp.}$ was always less than $P_{theor.}$, the difference being generally about 0.004 mm. Hg. As mentioned above, the velocity constant was always calculated from the smaller value $P_{exp.}$ At pressures greater than 0.04 mm. Hg this difference becomes negligible in the calculation of the velocity constant. At lower pressures the velocity constant calculated from $P_{theor.}$ becomes markedly smaller than that calculated from $P_{exp.}$ No definite explanation can be given for this difference. According to experiments at high pressures the N_2O_5 used was about 100 % pure. From all experimental indications and the character of the results obtained adsorption in the reaction vessel seems to be highly improbable. However, some N_2O_5 which did not decompose during the experiment may have been adsorbed on the walls of the unheated tubes connecting R with the valves and the Pirani gauge.

Summary.—The kinetics of the decomposition of N_2O_5 were investigated at pressures between 0.2 and 0.004 mm. Hg. The experiments show that there is no marked falling off of the constants down to pressures of

about 0.06 mm. Hg. At lower pressures, however, a decrease in the constants is to be found.

The theoretical discussion of this work will be published later.

The experiments described were carried out at the Institute of Physical Chemistry of the University of Berlin, to which institution the writers wish to acknowledge their indebtedness for the facilities furnished them.

* INTERNATIONAL RESEARCH FELLOW.

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*ADSORPTION OF AMMONIA BY PROTEINS**

BY WILDER D. BANCROFT AND C. E. BARNETT

BAKER LABORATORY OF CHEMISTRY, CORNELL UNIVERSITY

Communicated December 27, 1929

If we plot the amounts of ammonia taken up by unit mass of tartaric acid at constant temperature as abscissas and the equilibrium pressures, as ordinates, we get two flats, figure 1, the first corresponding to the formation of acid ammonium tartrate and the second to the formation of neutral ammonium tartrate. With succinic acid at ordinary temperature the two dissociation pressures are so nearly identical that we get what is apparently a single flat of double the length. At higher temperatures two flats are obtained with succinic acid. The simplest of the amino-acids, amino-acetic acid, takes up no ammonia; but this is not a necessary peculiarity of an amino-acid because *p*-aminobenzoic acid takes up ammonia just as one would expect it to.

In figure 2 is given the isotherm for casein and ammonia. Although casein takes up about 60 mg. of ammonia per gram of casein at 800 mm., there is no sign of any compound being formed. It is a clear case of adsorption. As is so generally the case with adsorption, there is a certain amount of hysteresis, the curve on addition of ammonia (circles) differing a little from that on removal of ammonia (dots). A curve similar to this was obtained by Mr. Tsunajima some years ago, while he was a graduate student at Cornell.

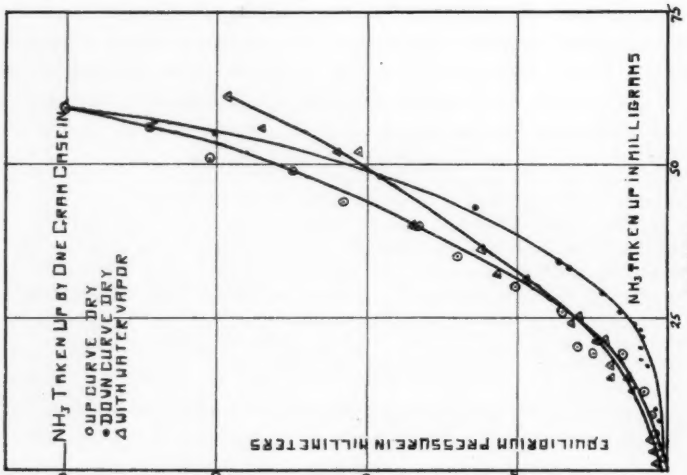

FIGURE 2

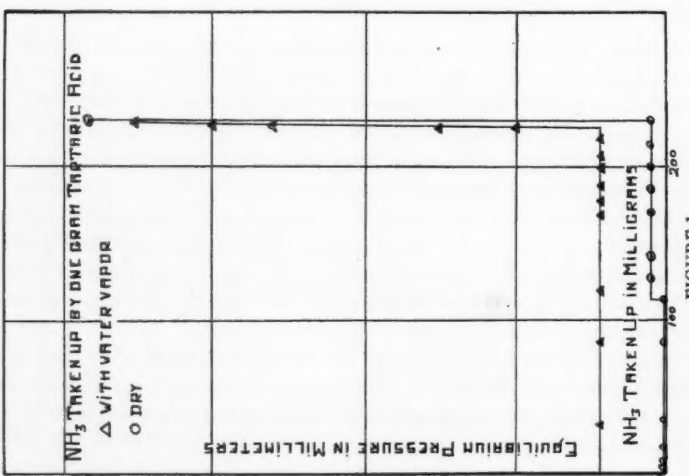

FIGURE 1

In figure 3 are given the data for gliadin and for fibrin and ammonia. In figure 4 are the corresponding data for arachin and for fibrin. The arachin, fibrin, gliadin and zein were presented to us by Professor R. A. Gortner, of the University of Minnesota, whom we thank for his courtesy. Mr. Belden, of the Cornell laboratory, has obtained an adsorption curve with no signs of compound formation for gelatin and ammonia. We are therefore justified in saying that ammonia is adsorbed by each of the six proteins, arachin, casein, fibrin, gelatin, gliadin, and zein; and that ammonia does not react in stoichiometrical proportions with any of them. Since ammonia is not taken up by glycine and since the amino-acids are probably weaker after condensation than before, it seems certain that few or none of the proteins will form definite chemical compounds with ammonia. However this may turn out, there are no solid ammonium caseinates or gelatinates at ammonia pressures up to one atmosphere.

The fact that ammonia does not form stoichiometric compounds with the proteins does not, of course, prove that the stronger bases such as caustic soda do not. Because there are no solid ammonium caseinates does not prove that there are no solid sodium caseinates. The present technique is only adapted to volatile acids and bases; but we are developing a new technique which will be more generally applicable. Since casein is not soluble in alcohol, for instance, it is possible to treat solid casein with alcoholic sodium hydroxide and to determine the amount of caustic soda removed from the alcohol by the solid casein. If the amounts taken up by the solid protein are plotted against the equilibrium concentrations, we shall get a flat in case there are two solid phases present, casein and a sodium caseinate, or two sodium caseinates. Theoretically, there is an error in this method because of the water formed during the reaction. This can be made as small as one pleases and will therefore be negligible. It could probably be eliminated completely by using sodium ethylate instead of sodium hydroxide; but the experimental difficulties would be so much greater as to make the advantage illusory.

A similar technique can of course be developed for acids. Since casein is insoluble in benzene, one could treat solid casein with a solution of acetic acid in benzene, for instance. No experiments of this sort have yet been made; but it is evident that practically any acid or alkali can be studied with any protein by making a suitable choice of a solvent.

The general results of this paper are as follows:

1. Ammonia gas combines stoichiometrically with tartaric acid, as everybody knew before.

2. Ammonia gas does not combine stoichiometrically with amino-acetic acid; but does with *p*-aminobenzoic acid.

3. Ammonia is adsorbed by, and does not combine stoichiometrically with arachin, casein, edestin, fibrin, gelatin, gliadin and zein.

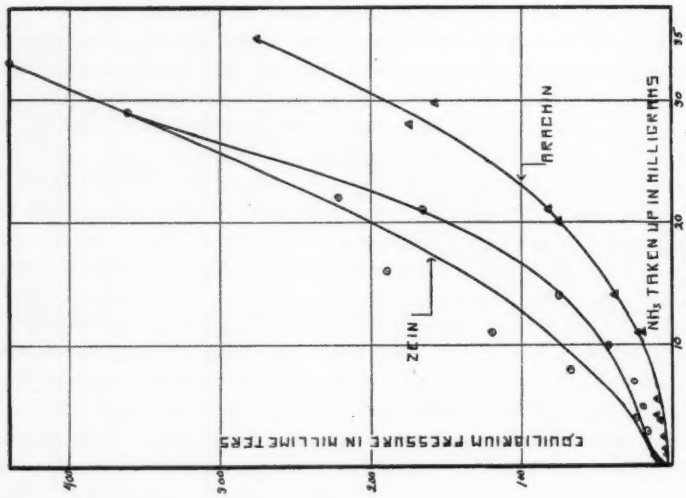

FIGURE 4

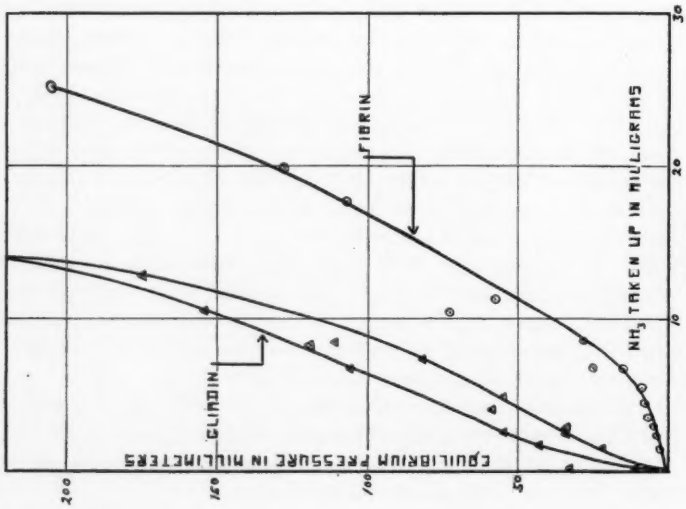

FIGURE 3

4. It is probable that ammonia combines stoichiometrically with very few of the proteins. A case of stoichiometrical combination might perhaps be found in a protein consisting chiefly of dibasic amino-acids, because ammonia does combine stoichiometrically with one of the carboxyl groups in glutamic acid.

5. The fact that there are no solid ammonium caseinates or ammonium gelatinates at room temperature does not prove that there are no solid sodium caseinates or sodium gelatinates.

6. By using a solvent which will dissolve sodium hydroxide or acetic acid but which will not peptize or dissolve gelatin, sodium gelatinate, or gelatin acetate, it should be possible to tell whether solid sodium gelatinate and solid gelatin acetate are possible stable phases.

* This work is part of the programme now being carried out at Cornell University under a grant from the Heckscher Foundation for the Advancement of Research established by August Heckscher at Cornell University.

DETERMINATION OF SOME PROPERLY IRREGULAR CYCLOTOMIC FIELDS

BY ELIZABETH T. STAFFORD AND H. S. VANDIVER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN
DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF TEXAS

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1. The integer h which represents the number of classes of ideals in a cyclotomic field defined by $e^{2i\pi/l}$, where l is an odd prime, can be written in the form h_1h_2, where h_1 and h_2 are both integers. If $h \equiv 0 \pmod{l}$ the field is called irregular. The necessary and sufficient condition that h_1 be divisible by l is that one of the first $(l-3)/2$ Bernoulli numbers be divisible by l.[1] A necessary, but not a sufficient, condition that h_2 be divisible by l is that h_1 be divisible by l.[2] A cyclotomic field in which h_1 is divisible by l, but h_2 is prime to l, is called a properly irregular cyclotomic field. By the use of the formulas derived in §2 it can be shown that, for $l < 211$, the only primes for which h_1 is divisible by l are $l = 37$, 59, 67, 101, 103, 131, 149, 157. By the methods outlined in §3 it can be demonstrated that h_2 is not divisible by l for any of these primes. It follows that the cyclotomic field defined by $e^{2i\pi/l}$ will be a properly irregular cyclotomic field when l assumes any of these values.

2. The problem of determining irregular cyclotomic fields reduces to that of investigating the divisibility of the Bernoulli numbers. This problem is simplified by the use of the formulas which will now be derived.

The congruence[3]

$$\frac{(1-n^i)b_i}{in^{i-1}} \equiv \sum_{u=1}^{n=l} \sum_{s=1}^{[ul/n]} s^{i-1} \pmod{l}, \tag{1}$$

where $[ul/n]$ denotes the largest integer contained in ul/n, is satisfied if

$$b_1 = -1/2, \quad b_{2a+1} = 0, \quad b_{2a} = (-1)^{a-1} B_a,$$

$B_1 = 1/6$, $B_2 = 1/30$, etc., being the Bernoulli numbers, $0 < i < l-1$. Define for $l > 6$

$$A_k = \sum_{s=[(k-1)l/4]+1}^{[kl/4]} s^{2a-1} \tag{2}$$

$$C_R = \sum_{s=[(k-1)l/6]+1}^{[kl/6]} s^{2a-1}, \tag{3}$$

where $[x]$ is the greatest integer in x.

Let $n = 2$ in (1)

$$\frac{(1-2^{2a})b_{2a}}{2a.2^{2a-1}} \equiv \sum_{s=1}^{[l/2]} s^{2a-1} \pmod{l}.$$

Using the relation $2^{l-1} \equiv 1 \pmod{l}$, this reduces to

$$\frac{(2^{l-2a}-2)b_{2a}}{2a} \equiv \sum_{s=1}^{[l/4]} s^{2a-1} + \sum_{s=[l/4]+1}^{[l/2]} s^{2a-1} \pmod{l}$$

or

$$\frac{(2^{l-2a}-2)b_{2a}}{2a} \equiv A_1 + A_2 \pmod{l}. \tag{4}$$

Let $n = 3$ in (1)

$$\frac{(1-3^{2a})b_{2a}}{2a.3^{2a-1}} \equiv \sum_{s=1}^{[l/3]} s^{2a-1} + \sum_{s=1}^{[2l/3]} s^{2a-1} \pmod{l}$$

$$\frac{(3^{l-2a}-3)b_{2a}}{2a} \equiv 2(C_1 + C_2) + C_3 + C_4 \pmod{l}. \tag{5}$$

Since we may take $l > 3$, then $l = 3k+1$ or $3k+2$. Consider

$$\left[\frac{l}{3}\right] + n = \left[l - \frac{2l}{3}\right] + n = l - 1 - \left[\frac{2l}{3}\right] + n$$

$$= l - \left(\left[\frac{2l}{3}\right] - \{n-1\}\right); n = 1, 2, \ldots, \left[\frac{l}{6}\right].$$

Then

$$\left(\left[\frac{l}{3}\right] + n\right)^{2a-1} \equiv \left(\left[\frac{2l}{3}\right] - \{n-1\}\right)^{2a-1} (-1)^{2a-1} \pmod{l},$$

$$\equiv -\left(\left[\frac{2l}{3}\right] - \{n-1\}\right)^{2a-1} \pmod{l}.$$

The number of terms in the sum $C_3 + C_4$ is even, since if $l = 3k + 1$, (k even),

$$\left[\frac{2l}{3}\right] - \left[\frac{l}{3}\right] = k$$

and if $l = 3k + 2$, (k odd),

$$\left[\frac{2l}{3}\right] - \left[\frac{l}{3}\right] = k + 1.$$

Then

$$\left(\left[\frac{l}{3}\right] + n\right)^{2a-1} + \left(\left[\frac{2l}{3}\right] - \{n-1\}\right)^{2a-1} \equiv 0 \pmod{l}.$$

It follows that

$$C_3 + C_4 \equiv 0 \pmod{l}.$$

Using this relation, (5) becomes

$$\frac{(3^{l-2a} - 3)b_{2a}}{2a} \equiv 2(C_1 + C_2) \pmod{l}. \qquad (5')$$

Let $n = 4$ in (1), we have

$$\frac{(4^{l-2a} - 4)b_{2a}}{2a} \equiv 3A_1 + 2A_2 + A_3 \pmod{l}. \qquad (6)$$

But

$$A_2 + A_3 = \sum_{s=[l/4]-1}^{[3l/4]} s^{2a-1}.$$

The number of terms in the sum $A_2 + A_3$ is even, since if $l = 4k + 1$,

$$\left[\frac{3l}{4}\right] - \left[\frac{l}{4}\right] = 3k - k = 2k$$

and if $l = 4k + 3$,

$$\left[\frac{3l}{4}\right] - \left[\frac{l}{4}\right] = 3k + 2 - k = 2k + 2.$$

Consider

$$\left[\frac{l}{4}\right] + n = (l-1) - \left[\frac{3l}{4}\right] + n; \quad n = 1, 2, \ldots, \left[\frac{l}{4}\right]$$

$$= l - \left(\left[\frac{3l}{4}\right] - \{n-1\}\right).$$

It follows that

$$\left(\left[\frac{l}{4}\right] + n\right)^{2a-1} + \left(\left[\frac{3l}{4}\right] - \{n-1\}\right)^{2a-1} \equiv 0 \pmod{l}$$

and (6) becomes, since

$$A_2 + A_3 \equiv 0 \pmod{l}$$

$$\frac{(4^{l-2a} - 4)b_{2a}}{2a} \equiv 3A_1 + A_2 \pmod{l}. \qquad (6')$$

Finally, let $n = 6$ in (1). The result may be written

$$\frac{(6^{l-2a} - 6)b_{2a}}{2a} \equiv 5C_1 + 4C_2 + 3C_3 + 2C_4 + C_5 \pmod{l}. \qquad (7)$$

In C_2, the number of terms is given by $[l/3] - [l/6]$. If $l = 6k + 1$,

$$\left[\frac{l}{3}\right] - \left[\frac{l}{6}\right] = 2k - k = k$$

and if $l = 6k + 5$,

$$\left[\frac{l}{3}\right] - \left[\frac{l}{6}\right] = 2k + 1 - k = k + 1.$$

In C_5 the number of terms is given by $[5l/6] - [2l/3]$.
If $l = 6k + 1$,

$$\left[\frac{5l}{6}\right] - \left[\frac{2l}{3}\right] = 5k - 4k = k$$

and if $l = 6k + 5$,

$$\left[\frac{5l}{6}\right] - \left[\frac{2l}{3}\right] = 5k + 4 - 4k - 3 = k + 1.$$

The number of terms in C_2 equals the number of terms in C_5, and

$$\left(\left[\frac{l}{6}\right] + n\right)^{2a-1} = \left(l - \left[\frac{5l}{6}\right] + \{n-1\}\right)^{2a-1} \equiv -\left(\left[\frac{5l}{6}\right] - \{n-1\}\right)^{2a-1} \pmod{l}, \quad n = 1, 2, \ldots \left[\frac{l}{6}\right].$$

It follows that

$$C_2 + C_5 \equiv 0 \pmod{l}.$$

Then (7) becomes

$$\frac{(6^{l-2a} - 6)b_{2a}}{2a} \equiv 5C_1 + 3C_2 + C_3 \pmod{l}. \qquad (7')$$

From the definitions of A_k and C_k, it is evident that

$$A_1 + A_2 = C_1 + C_2 + C_3 = \sum_{s=1}^{[l/2]} s^{2a-1}. \tag{8}$$

From (7′)

$$\frac{(6^{l-2a} - 6)b_{2a}}{2a} \equiv 2(C_1 + C_2) + 2C_1 + A_1 + A_2 \pmod{l}$$

$$\equiv \frac{3^{l-2a} - 3}{2a} b_{2a} + \frac{4^{l-2a} - 4}{2a} b_{2a} + 2(C_1 - A_1) \pmod{l}$$

or

$$2(A_1 - C_1) \equiv \frac{(3^{l-2a} - 3) + (4^{l-2a} - 4) - (6^{l-2a} - 6)}{2a} b_{2a} \pmod{l}.$$

In this congruence substitute for $A_1 - C_1$ and b_{2a} the values given by (2) and (3), we obtain for $2a < l - 1$, for any odd l, since it obviously holds for $l = 5$,

$$\sum_{s=[l/6]+1}^{[l/4]} s^{2a-1} \equiv \frac{1 - 3^{l-2a} - 4^{l-2a} + 6^{l-2a}}{4a} (-1)^a B_a \pmod{l}, \tag{9}$$

a result given without proof by Mirimanoff[4] for the special case where $l - 1$ is divisible by 12.

Also from (7′)

$$\frac{(6^{l-2a} - 6)}{2a} b_{2a} \equiv 2\{2(C_1 + C_2)\} + A_1 + A_2 - 2C_2 \pmod{l}$$

$$\equiv \frac{2(3^{l-2a} - 3) + 2^{l-2a} - 2}{2a} b_{2a} - 2C_2 \pmod{l}$$

or

$$C_2 \equiv \frac{(-1)^a B_a}{4a} (2.3^{l-2a} - 6 + 2^{l-2a} - 2 - 6^{l-2a} + 6) \pmod{l}.$$

Then

$$\sum_{s=[l/6]+1}^{[l/3]} s^{2a-1} \equiv \frac{(-1)^a B_a (2^{l-2a-1} - 1)(3^{l-2a} - 1)}{2a} \pmod{l}. \tag{10}$$

From (6′)

$$\frac{(3^{l-2a} - 3)}{2a} b_{2a} \equiv 2(C_1 + C_2) \pmod{l}$$

$$\equiv 2(A_1 + A_2) - 2C_3 \pmod{l}$$

$$\equiv 2\frac{(2^{l-2a} - 2)b_{2a}}{2a} - 2C_3 \pmod{l}$$

or

$$C_3 \equiv \frac{2^{l-2a+1} - 4 - 3^{l-2a} + 3}{4a} (-1)^{a-1} B_a \pmod{l}.$$

This gives

$$\sum_{s=[l/3]+1}^{[l/2]} s^{2a-1} \equiv \frac{2^{l-2a+1} - 3^{l-2a} + 1}{4a} (-1)^{a-1} B_a \pmod{l}. \quad (11)$$

Finally

$$\sum_{s=1}^{l-1} s^{2a-1} = 1^{2a-1} + 3^{2a-1} + \ldots + (l-2)^{2a-1} + 2^{2a-1} \sum_{s=1}^{[l/2]} s^{2a-1}.$$

Also

$$A_1 + A_2 + A_3 + A_4 - 2^{2a-1}(A_1 + A_2) \equiv 1^{2a-1} + 3^{2a-1} + \ldots + (l-2)^{2a-1} \pmod{l}.$$

But

$$A_2 + A_3 \equiv 0 \pmod{l} \text{ and } A_1 + A_4 \equiv 0 \pmod{l}.$$

Therefore from (4),

$$-2^{2a-1} \frac{(1 - 2^{2a}) b_{2a}}{2a \cdot 2^{2a-1}} \equiv 1^{2a-1} + 3^{2a-1} + \ldots + (l-2)^{2a-1} \pmod{l}$$

and

$$\frac{2^{2a} - 1}{2a} (-1)^{a-1} B_a \equiv 1^{2a-1} + 3^{2a-1} + \ldots + (l-2)^{2a-1} \pmod{l} \quad (12)$$

The formula (9) was applied for all values of $l < 211$. In case the left member of the congruence was congruent to zero modulo l, the factor $1 - 4^{l-2a} - 3^{l-2a} + 6^{l-2a}$ was investigated. If it was not congruent to zero, then B_a is divisible by l. As stated in §1, this condition was satisfied only for $l = 37, 59, 67, 101, 103, 131, 149, 157$. In case the factor $1 - 4^{l-2a} - 3^{l-2a} + 6^{l-2a}$ was congruent to zero, the divisibility of B_a was still indeterminate and one of the formulas (10), (11) or (12) was applied. For values of l between 100 and 210, (9) established the divisibility or non-divisibility of the Bernoulli numbers with the exception of the cases listed here.

(*a*) $l = 127$ and $a = 18, 25, 32, 36, 39, 46, 53, 60$. Formula (11) was applied for these values of a and in each case the left member was not congruent to zero modulo 127.

(*b*) $l = 137$ and $a = 29$. Formula (10), when applied for $a = 29$ was not congruent to zero modulo 137.

(*c*) $l = 151$ and $a = 23, 38, 53, 68$. These cases were all settled by formula (11).

(*d*) $l = 167$ and $a = 42$. This was settled by formula (12) which had its right member not congruent to zero modulo 167.

(*e*) $l = 181$ and $a = 47$. This was settled by formula (11).

(*f*) $l = 191$ and $a = 25, 48, 68$. Formula (11) was used for $a = 25$, and 68, but formula (12) was necessary for $a = 48$.

(*g*) $l = 193$ and $a = 48$. Here (9), (10), (11) and (12) all vanish modulo l, but we find $B_{48} \not\equiv 0 \pmod{193}$ using actual value of B_{48} from Adams' tables.

(*h*) $l = 197$ and $a = 51$. This was settled by (11).

(*i*) $l = 199$ and $a = 50$. This was settled by formula (11).

The computation involved in the application of these formulas was carried through by the use of a table of indices such as that in Cahen, *Theorie des Nombres*, vol. 2, p. 39. For $l = 127$ the part of the computation with $a = 17$, 18 and 19 is indicated below.

S	I	$2I$	$33I$	N	$35I$	N	$37I$	N
22	70	14	42	19	56	52	70	22
23	11	22	111	5	7	105	29	46
24	77	28	21	26	49	90	77	24
25	96	66	18	2	84	107	24	73
26	38	76	120	47	70	22	20	13
27	69	12	9	111	21	20	33	102
28	35	70	21	20	91	59	35	28
29	79	32	87	54	119	75	25	83
30	26	52	102	87	28	68	80	113
31	50	100	12	94	112	37	86	124
				459		635		628

Since $[127/6] + 1 = 22$ and $[127/4] = 31$, the first column contains the values of s to be used for each value of a. The second column (headed I) contains the indices of the numbers as determined from the table of indices for $l = 127$. The next column (headed $2I$) is obtained by multiplying each of the numbers in the second column by 2 and then reducing modulo 126. If $a = 17$, $2a - 1 = 33$ and the column headed $33I$ contains the residues modulo 126 of s^{33} for $s = 22, 23, \ldots, 31$. Add to each of these numbers the number of the $2I$ column which appears in the same row and reduce modulo 126. The resulting numbers will be those appearing in the column headed $35I$. A repetition of this process gives the numbers appearing in the $37I$ column. In this manner the columns for $a = 12, 13, \ldots, (l-3)/2$ (or for $2a - 1 = 23, 24, \ldots, l - 4$) were computed. The column headed N, which follows each kI column, was filled in by finding, in the table of indices and numbers, the number which had the index given in the kI column. These N columns were summed and reduced modulo 127. For $a = 18$ (or $2a - 1 = 35$), the N column sums to $635 \equiv 0 \pmod{127}$. Since the left member of (9) is congruent to zero, the factor $1 - 4^{l-2a} - 3^{l-2a} + 6^{l-2a}$ must be investigated for $l = 127$ and $a = 18$. $l - 2a = 91$ and $1 - 4^{91} - 3^{91} + 6^{91} \equiv 1 - 1 - 24 + 24 \equiv 0 \pmod{127}$. Formula (9) does not decide the problem of the di-

visibility of B_{18} by 127. Using the method outlined above with formula (11) and $a = 18$, the left member of (11) is not congruent to zero modulo 127 so that $B_{18} \not\equiv 0 \pmod{127}$.

3. In order to show that the cyclotomic fields defined by for $l = 37$, 59, 67, 101, 103, 131, 149, 157 are properly irregular cyclotomic fields, it must be proved that h_2 is prime to l for each of these values. The divisibility of h_2 by l can be investigated by means of certain properties of sets of independent units in the field defined by $e^{2i\pi/l}$. Kummer[5] proves that the set of units $e(\alpha), e(\alpha^\gamma), \ldots, e(\alpha^{\gamma\mu-2})$; $\left[\begin{array}{l}\alpha = e^{2i\pi/l}; \mu = \dfrac{l-1}{2} \\ \gamma = \text{primitive root of } l\end{array}\right]$

where

$$e(\alpha) = \sqrt{\frac{(1-\alpha^\gamma)(1-\alpha^{-\gamma})}{(1-\alpha)(1-\alpha^{-1})}} = \alpha^{1-\gamma/2}\frac{\alpha^\gamma - 1}{\alpha - 1} \tag{13}$$

forms an independent set. Hence there exists a certain integral power of every unit in the field such that the unit raised to that power can be expressed as the product of integral powers of these independent units.

Let $\epsilon_1(\alpha), \epsilon_2(\alpha), \ldots, \epsilon_{\mu-1}(\alpha)$ form a fundamental set of real units.[6] They satisfy the following regulations.

$$\left\{\begin{array}{l}\epsilon_1(\alpha)^{n_1} = e(\alpha)^{r_1^1}.e(\alpha^\gamma)^{r_2^1} \ldots\ e(\alpha^{\gamma^{\mu-2}})^{r_{\mu-1}^1} \\ \epsilon_2(\alpha)^{n_2} = e(\alpha)^{r_1^2}\, e(\alpha^\gamma)^{r_2^2} \ldots\ e(\alpha^{\gamma^{\mu-2}})^{r_{\mu-1}^2} \\ \epsilon_{\mu-1}(\alpha)^{n_{\mu-1}} = e(\alpha)^{r_1^{\mu-1}}\, e(\alpha^\gamma)^{r_2^{\mu-1}} \ldots\ e(\alpha^{\gamma^{\mu-2}})^{r_{\mu-1}^{\mu-1}},\end{array}\right. \tag{14}$$

where each n_k has no factor in common with all of the integers r_1^k, r_2^k, r_3^k, $\ldots\ r_{\mu-1}^k$; the parentheses being omitted in the superscripts of the r's. Form

$$\left\{\begin{array}{l}n_1 l\epsilon_1(\alpha) = r_1^1 le(\alpha) + r_2^1 le(\alpha^\gamma) + \ldots + r_{\mu-1}^1 le(\alpha^{\gamma^{\mu-2}}) \\ n_2 l\epsilon_2(\alpha) = r_1^2 le(\alpha) + r_2^2 le(\alpha^\gamma) + \ldots + r_{\mu-1}^2 le)\alpha(^{\gamma^{\mu-2}}) \\ n_{\mu-1}\, l\epsilon_{\mu-1}(\alpha) = r_1^{\mu-1}\, le(\alpha) + r_2^{\mu-1}\, le(\alpha^\gamma) + \ldots + r_{\mu-1}^{\mu-1} le(\alpha^{\gamma^{\mu-2}})\end{array}\right. \tag{15}$$

where $l\epsilon_k(\alpha)$ stands for the real value of the logarithm of $\epsilon_k(\alpha)$, etc. Now define D, Δ and R by the following determinants.

$$D = \left|\begin{array}{l} le(\alpha)\ le(\alpha^\gamma)\ le(\alpha^{\gamma 2}) \ldots\ldots\ldots\ldots\ le(\alpha^{\gamma\mu-2}) \\ le(\alpha^\gamma)\ le(\alpha^{\gamma 2}\ le(\alpha^{\gamma 3}) \ldots\ldots\ldots\ldots\ le(\alpha^{\gamma\mu-1}) \\ le(\alpha^{\gamma\mu-2})\ le(\alpha^{\gamma\mu-1})\ le(\alpha^{\gamma\mu}) \ldots\ldots\ le(\alpha^{\gamma 2\mu-4})\end{array}\right| \tag{16}$$

$$\Delta = \begin{vmatrix} l\epsilon_1(\alpha) & l\epsilon_2(\alpha) & \dots & l\epsilon_{\mu-}(\alpha) \\ l\epsilon_1(\alpha^\gamma) & l\epsilon_2(\alpha^\gamma) & \dots & l\epsilon_{\mu-1}(\alpha^\gamma) \\ l\epsilon_1(\alpha^{\gamma^{\mu-2}}) & l\epsilon_2(\alpha^{\gamma^{\mu-2}}) & \dots & l\epsilon_{\mu-1}(\alpha^{\gamma^{\mu-1}}) \end{vmatrix} \tag{17}$$

$$R = \begin{vmatrix} r_1^1 & r_2^1 & \dots & r_{\mu-1}^1 \\ r_1^2 & r_2^2 & \dots & r_{\mu-1}^2 \\ r_1^{\mu-1} & r_2^{\mu-1} & \dots & r_{\mu-1}^{\mu-1} \end{vmatrix} \tag{18}$$

If we form $R \cdot D$ and actually carry out the multiplication of the determinants, using the multiplication theorem, and then substitute the values given by (15) where α is replaced successively by α, α^γ, α^{γ^2}, ..., $\alpha^{\gamma^{\mu-2}}$, we obtain the relation

$$R \cdot D = n_1 n_2 \dots n_{\mu-1} \Delta. \tag{19}$$

But, $R \neq 0$ is a necessary condition for an independent set of units,[7] so that (19) may be replaced by

$$D/\Delta = \frac{n_1 n_2 \dots n_{\mu-1}}{R}. \tag{20}$$

The expression D/Δ is the second factor of the class number.[8] It is evident that if D/Δ is to be divisible by l, then at least one of the numbers $n_1, n_2 \dots n_{\mu-1}$ must be divisible by l. From this it follows that there exists a unit $\epsilon_a(\alpha)$ such that

$$(\epsilon_a(\alpha))^{lg} = e(\alpha)^{r_1} e(\alpha^\gamma)^{r_2} \dots e(\alpha^{\gamma^{\mu-2}})^{r_{\mu-1}}, \tag{21}$$

where not all of the numbers $r_1, r_2, \dots, r_{\mu-1}$ are divisible by l. We have easily

$$e(\alpha^{\gamma^i})^\mu = E_1^{\gamma^{2i}} E_2^{\gamma^{4i}} \dots E_{\mu-1}^{\gamma^{2i(\mu-1)}} \beta^l$$

β, a unit in $k(\alpha)$; $i = 0, 1, 2, \dots, \mu - 1$.

Substituting in (21) gives

$$\epsilon_a(\alpha)^{lg\mu} = E_1^{t_1} E_2^{t_2} \dots E^{t_{\mu-1}} \beta_1^l$$

where

$$t_i = r_1 + r_2\gamma^{2i} + \dots + r_{\mu-1}\gamma^{2i(\mu-2)}$$
$$i = 1, 2, \dots, \mu - 1$$

Now, not all the t's are divisible by l for if so we obtain

$$t_i \equiv 0 \pmod{l}; \quad i = 1, 2, \dots, \mu - 1.$$

Putting this in terms of the r's we get $\mu - 1$ congruences from which we may eliminate γ and obtain

$$r_i \equiv 0 \pmod{l};$$
$$i = 1, 2, \ldots, \mu - 1;$$

since the determinant formed by the coefficients of the r's is prime to l, but this contradicts the hypothesis that not all these r's have a common factor. Hence not all the t's have a common factor and by using a result that is established elsewhere[9] it follows that we have

$$E_i = \delta^l$$

for at least one value of i. As shown in another paper[10] it follows that $B_i \equiv 0 \pmod{l}$. Hence if $B_n \equiv 0 \pmod{l}$ and we can show that $E_n \neq \delta_1^l$, where δ_1 is a unit in $k(\alpha)$, then it follows that $k(\alpha)$ is a properly irregular cyclotomic field for that value of l. To effect this we select the smallest national prime q such that $q \equiv 1 \pmod{l}$. Then q decomposes in $k(\alpha)$ into the product of $l - 1$ distinct ideal factors. Using Cahen's and Jacobi's tables of indices (and others constructed by the authors) for the prime q we select the rational integer d whose index is $(q - 1)/l$. Consider the ideal $P = (\alpha - d, q)$ which is a prime ideal factor of q. From this we obtain $\alpha \equiv d \pmod{P}$. If we represent by $E_n(d)$ the number obtained by substituting d for α in E_n we derive, from $E_n = \delta_1^l$,

$$E_n(d) \equiv c_1^l \pmod{q},$$

where c is a rational integer. We then reduce $E_n(d)$ by the table of indices and determine if ind. $E_n(d) \equiv 0 \pmod{l}$. If it is not, then $k(\alpha)$ is a properly irregular cyclotomic field.

The table below gives the values of n, γ, d, q for each irregular $l < 211$. The last column, headed ind. $E_n(d)$ gives the residue of ind. $E_n(d)$, modulo l in each case.

l	n	γ	d	q	IND. $E_n(d)$
37	16	2	17	149	24
59	22	2	385	709	50
67	29	2	47	269	4
101	34	2	100	809	45
103	12	5	315	619	65
131	11	2	100	263	52
149	65	10	354	1093	127
157	31	139	1024	1571	150
157	55	139	1024	1571	39

Tables of indices for the primes 709, 269, 809, 619 and 263 are found in the "Canon Arithmeticus," C. G. J. Jacobi (1839), but tables of indices had to be constructed for the primes 1193 and 1571, the former to the base 10 and the latter to the base 2.

As the table shows, the computation established the fact that $E_n(\alpha)$ is not the lth power of a unit of the field for any of these cases. It follows that the cyclotomic fields defined by $e^{2i\pi/l}$ where l takes the values 37, 59, 67, 101, 103, 131, 149, 157 are properly irregular cyclotomic fields.

All the calculations mentioned above were carried out by Miss Stafford with the exception of those relating to the proof that the second factor of the class number is prime to l for $l = 157$, the latter having been carried out by Mr. Vandiver. Miss Stafford made all her computations concerning the second factor of the class number for the various primes l in two different ways. The computations of Mr. Vandiver concerning the case $l = 157$ were all checked by Mr. S. S. Wilks, tutor in mathematics, University of Texas. For the case $l = 37$, Mr. Vandiver went through the computations concerning the second factor of the class number as a pattern for his work on the case $l = 157$. He thereby checked again the computations of Kummer and Miss Stafford for $l = 37$.

The definition that was given at the beginning of this paper of a properly irregular cyclotomic field is equivalent to that which was given for such a field in another article,[11] as it will be proved in another paper[12] that a necessary and sufficient condition that the second factor of the class number of a cyclotomic field defined by an odd prime l be divisible by l is that one of the units $E_i(\alpha)$ is the lth power of a unit in a field; $i = 1, 2, \ldots, (l - 3)/2$.

4. In the determination of the irregular primes only the data was mentioned above concerning the computation for the primes $100 < l < 211$. Using the process above described, Mrs. A. C. S. Williams checked all the primes < 100, and found only the irregular primes 37, 59 and 67 as had Kummer. Miss Stafford's computations as to the irregular primes check with Kummer's conclusion that the only irregular primes $100 < l < 167$ are 101, 103, 131, 149 and 157. She found in each case the Bernoulli numbers divisible by l which Kummer had found. Her work concerning the divisibility of the second factor of the class number of $k(\alpha)$ by l for $l = 37$, 59 and 67 agrees with the results of Kummer's computations on the same.[13]

The papers containing all the above-mentioned computations are now in the possession of Mr. Vandiver at the University of Texas; ultimately, they will probably be deposited in some University library.

[1] Kummer, *J. Math.*, **16**, 1851, p. 479. A fraction in its lowest terms is said to be divisible by l if its numerator is divisible by l.

[2] Kummer, loc. cit., p. 486.

[3] Vandiver, *Ann. Math.*, **18**, 1917, p. 114.

[4] *Crelle*, **128**, 1905, pp. 45–68.

[5] *J. Math.*, **16**, 1851, p. 396.

[6] Kummer, *Ibid*, **16**, 1851, p. 399.

[7] *Ibid*, **16**, 1851, p. 387.

[8] Kummer, *J. Math.*, **16**, 1851, p. 471.

[9] Vandiver, *Bull. Amer. Math. Soc.*, **35**, 1929, 333–35.

[10] Transactions, *Amer. Math. Soc.*, **31**, 1929, pp. 628–9. The result is obtained by treating the relation $E_i = \delta^i$ in the same way that relation (15) p. 628 is treated.

[11] Vandiver, *Proc. Nat. Acad. Sci.*, **15**, 1929, p. 202.

[12] Vandiver, *Ibid.*, **16**, 1930.

[13] Berlin Abhandlungen, *Math. Phys. Klasse*, 1857, pp. 73–74.

ON AN EXTENSION OF THE FOURIER THEOREM GIVING RAPID METHODS FOR CALCULATING THE CONSTANT PART AND THE COEFFICIENT OF ANY PERIODIC TERM IN THE DISTURBING FUNCTION

By Ernest W. Brown

Departments of Mathematics and Astronomy, Yale University

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1. Let $F(X)$, $G(X)$ be functions which may be expanded into cosines (or sines) of angles of the form $iX + b$, where i, b are constants. Let w be another variable related to X by means of the equation,

$$w = X + G_1(X). \tag{1.1}$$

Then $F(X)$ may be expressed in terms of w by means of Lagrange's theorem

$$F(X) = F - G\cdot F_1 + \frac{1}{2!}\frac{d}{dw}(G^2\cdot F_1) - \frac{1}{3.!}\frac{d^2}{dw^2}(G^3\cdot F_1) + \ldots, \tag{1.2}$$

where the notation,

$$F = F(w),\ F_1 = \frac{d}{dw}F(w),\ G = G(w),\ G_1 = \frac{d}{dw}G(w),$$

is used, and where it is supposed that G contains a small parameter as factor such that the series (1.2) can be used for calculation.

Let

$$H_1 \equiv H_1(w) \equiv \frac{d}{dw}H(w) \equiv \frac{dH}{dw}$$

be a function of the same nature as G and F, i.e., it is expressible as a sum of cosines (or sines) of angles of the form $iw + b$. In the applications, H_1 contains no constant term, so that H contains no term of the form kw, but it will be shown that the result obtained is independent of the presence of such a term in H.

We have, if $D \equiv d/dw$,

$$H_1D^n\phi = D(H_1D^{n-1}\phi) - DH_1\cdot D^{n-1}\phi.$$

If ϕ be a function of w of the same nature as H_1, the first term of the right-hand member is still a function of the same kind but contains no constant term. Hence the constant term of the second term of the right-hand member is the same as that of the left-hand member. By following a similar procedure n times, we reach the conclusion that the two expressions

$$H_1\cdot D^n\phi, \quad (-1)^n\phi\cdot D^nH_1 \equiv (-1)^n\phi\cdot H_{n+1}$$

have the same constant term.

If we multiply (1.2) by H_1 and apply this result to each of the terms we obtain the result that the constant term in the expansion of $H_1(w)\cdot F(X)$ is the same as that of

$$H_1\cdot F - F_1(H_1\cdot G + \frac{1}{2!}H_2\cdot G^2 + \frac{1}{3!}H_3\cdot G^3 + \ldots). \qquad (1.3)$$

But by Taylor's theorem,

$$H(w + G) = H + H_1\cdot G + \frac{1}{2!}H_2\cdot G^2 + \ldots,$$

so that (1.3) may be written,

$$H_1\cdot F + F_1\cdot H - F_1\cdot H(w + G). \qquad (1.4)$$

Further, we have

$$D\{F\cdot H(w + G)\} = F_1\cdot H(w + G) + F\cdot(1 + G_1)\cdot H_1(w + G),$$

and $H_1\cdot F + F_1\cdot H = D(F\cdot H)$, so that (1.4) becomes

$$D\{F\cdot H - F\cdot H(w + G)\} + F\cdot(1 + G_1)\cdot H_1(w + G). \qquad (1.5)$$

Since $H - H(w + G)$ contains only constant and periodic terms, the first term of (1.5) contains no constant term. Hence,

The constant term in the expansion of $H_1(w)\cdot F(X)$, when this expression is transformed to a sum of cosines (or sines) of angles of the form $iw + b$ by means of the relation $w = X + G(X)$, is the same as the constant term in the expansion of

$$H_1(w + G)\cdot\left(1 + \frac{dG}{dw}\right) F(w), \quad G \equiv G(w),$$

and the constant term of the expansion of $F(X)$ is the same as that in

$$\left(1 + \frac{dG}{dw}\right) F(w).$$

When w, G take the values 0, π together, the result can be deduced from the Fourier theorem. The form of statement given above is, how-

ever, much more convenient in the applications than that of Fourier and it admits of easy generalization to any number of variables.

2. The theorem will be used here to obtain the coefficient of any term in the expansion of a function into a sum of periodic terms.

Suppose that we know in advance that $F(X)$ can be expanded into a sum of cosines of angles of the form $iw + b$ (i is not necessarily an integer) and that we desire the coefficient of $\cos(iw + b)$ in this expansion. This coefficient is evidently twice the constant term of the product $F(X)\cdot\cos(iw + b)$, when this product is expressed as a sum of periodic terms. The theorem therefore states that

The coefficient of $\cos(iw + b)$ *in the expansion of* $F(X)$ *when* $F(X)$ *is transformed to a function of* w *by means of the relation* $w = X + G(X)$ *is the constant term in the expansion of*

$$2\cos\{iw + b + iG(w)\}\cdot F(w)\cdot\{1 + \frac{d}{dw}G(w)\}, \tag{2.1}$$

or the constant term in

$$2\cos\{iX + b + iG(X)\}\cdot F(X)\cdot\{1 + \frac{d}{dX}G(X)\}, \tag{2.2}$$

or that in

$$-2\sin\{iX + b + iG(X)\}\frac{1}{i}\frac{d}{dX}F(X), \tag{2.3}$$

and the constant term is the same as that in

$$F(X)\cdot\{1 + \frac{d}{dX}G(X)\}. \tag{2.4}$$

The form (2.2) gives the same result as (2.1) because it makes no difference whether we use the letter w or X in finding the constant term. The form (2.3) is deduced from (2.2) by making use of the fact that (2.2) is equal to twice

$$\frac{1}{i}\frac{d}{dX}[F(X)\cdot\sin\{iX + b + iG(X)\}] - \frac{1}{i}\frac{d}{dX}F(X)\cdot\sin\{iX + b + iG(X)\},$$

and that the former of these two terms has no constant part.

3. The theorem can be extended to two or more variables. Stated for the number needed, namely, two, it reads.

The coefficient of the term $\cos(iw + i'w' + b)$ *in the expansion of* $F(X, X')$ *as a function of* w, w', *by means of the relations*

$$w = X + G(X), \quad w' = X' + G'(X'), \tag{3.1}$$

is the constant term in

$$2\cos N \cdot F(X, X') \cdot \left\{1 + \frac{d}{dX} G(X)\right\}\left\{1 + \frac{d}{dX'} G'(X')\right\}, \quad (3.2)$$

as in

$$-\frac{2}{ii'} \cos N \cdot \frac{\partial^2}{\partial X \partial X'} F(X, X'), \quad (3.3)$$

where

$$N \equiv iX + i'X' + b + iG(X) + i'G'(X'),$$

and the constant term is the same as that in

$$F(X, X') \cdot \left\{1 + \frac{d}{dX} G(X)\right\} \cdot \left\{1 + \frac{d}{dX'} G'(X')\right\}. \quad (3.4)$$

The deduction of (3.3) from (3.2) is a second application, for the variable X', of the deduction of (2.3) from (2.2).

4. The application of the theorem of §3 to the expansion of the disturbing function is made as follows. Suppose that the expansion has been made in terms of the eccentric anomalies X, X' and that we wish to transform to the true anomalies w, w'. The relations are

$$w = X - e \sin X, \quad w' = X' - e' \sin X',$$

where e, e' are the eccentricities.

If the expansion of the disturbing function in terms of X, X' be denoted by $R(X, X')$, the theorem says that the coefficient of $\cos(iw + i'w' + b)$ in the expansion of R in terms of w, w' is the constant term in

$$-\frac{2}{ii'} \cos N \cdot \frac{\partial^2}{\partial X \partial X'} R(X, X'), \quad (4.1)$$

where

$$N = iX + i'X' + b - ie \sin X - i'e' \sin X',$$

and that the part of R independent of w, w' will be the constant term in

$$(1 - e \cos X)(1 - e' \cos X')\, R(X, X'), \quad (4.2)$$

or in Rrr'/aa', since $r = a(1 - e \cos X)$, $r' = a'(1 - e' \cos X')$, where r, r' are the radii vectors and $2a$, $2a'$, $(1e' \cos X')$ the major axes of the two elliptic orbits.

In a previous paper,[1] I have shown how R can be expanded in terms of the eccentric anomalies by means of a general formula which enables us to get the coefficient of any term to any order whatever. This theorem shows how to extend the result to the mean anomalies. For the construction of the product, we put $b = 0$ for cosines and $b = \pi/2$ for sines and write

$$\cos N = \cos iN_0 \cos i'N_0' - \sin iN_0 \sin i'N_0',$$

where $N_0 = X - e \sin X$, $N_0' = X' - e' \sin X'$. Each of the four terms may be expressed as Fourier series with arguments X or X' by well-known formulae with Bessel coefficients.

We first form the products of $\partial^2 R/\partial X \partial X'$ by $\cos iN_0$, $\sin iN_0$ choosing out those portions which are independent of X, and then the products of the results by $\cos i'N_0'$, $\sin i'N_0'$, choosing out the portions independent of X'. It is to be noted that only terms in which w, w' are both present occur in the derivative of R.

It is not certain whether this method is more efficient than a direct change to the mean anomalies for the *general* development of R. Its chief value is that it enables us to obtain the coefficient of a particular term to any degree of accuracy required, and this additional accuracy is always needed for the computation of the terms of long period.

For the constant term of R, we form the simple products of (4.2) in the same way. Or we may develop Rr' directly in terms of the eccentric anomalies, since the formula I gave is applicable to Rr' as easily as to R. We then get the result:

The constant term in the development of R in terms of the mean anomalies is the same as the constant term of the development of Rr'/aa' in terms of the eccentric anomalies.

5. The theorems can be equally well applied when the initial development of R is made in terms of the *true* anomalies f, f'. This development can be made by the method of my former paper[1] and it is rather more simple because R is expressed at the outset as a function of r, r', f, f' and it is only necessary to expand it in terms of f, f' by means of the substitutions

$$\frac{a(1-e^2)}{r} = 1 + e\cos f, \quad \frac{a'(1-e'^2)}{r'} = 1 + e'\cos f'.$$

The relation between f, w is given by,

$$\frac{dw}{df} = \frac{r^2}{a^2(1-e^2)^{1/2}} = \frac{(1-e^2)^{3/2}}{(1+e\cos f)^2}$$

which, on integration, gives the series

$$w = f - 2e\sin f + {}^3/_4 e^2 \sin 2f \ldots \equiv f + G(f).$$

All the letters are accented to obtain the relation between w', f'.

The plan of §4 is now followed. The coefficients of the series for $\cos iN_0, \ldots$, are, however, no longer given by known functions, but must be expanded in powers of e, e'. The theorem for the constant part of R becomes:

The constant term of R developed in terms of the mean anomalies is the same as the constant term of

$$\frac{r^2r'^2}{a^2a'^2}(1-e^2)^{-1/2}(1-e'^2)^{-1/2}R,$$

developed in terms of the true anomalies.

6. Finally the theorem is applicable whether a literal or numerical development be undertaken. We can also replace R by its derivatives with respect to the elements or to the coördinates in the formulae. An interesting point with reference to the elliptic developments is the fact that they can all be made by direct harmonic analysis: the solution of Kepler's equation, $w = X - e \sin X$, to find X in terms of w, is avoided altogether and is replaced by the calculation of functions of w in terms of X or f.

7. As an example of the ease with which the theorem of §2 gives the coefficients in expansions which are sometimes quite troublesome, let us apply it to the expansion of

$$(1-e^2)^{1/2}(1-e\cos X)^{-2} \tag{7.1}$$

into a Fourier series with argument w where

$$w = X - e\sin X.$$

Since $dw/dX = 1 - e\cos X$, the theorem with the form (2.2) shows that the coefficient of $\cos iw$ in the required expansion is the constant term in the expansion of

$$2\cos(iX - ie\sin X)\cdot(1-e^2)^{1/2}(1-e\cos X)^{-1}$$

as a Fourier series with argument X.

If $J_p(x)$ be a Bessel function of the first kind, it is well known that

$$\cos(iX - ie\sin X) = \Sigma_p J_p(ie)\cos(i-p)X, \quad p = 0, \pm 1, \pm 2, \ldots$$

Also if $e = 2\eta/(1+\eta^2)$, we have

$$(1-e^2)^{1/2}(1-e\cos X)^{-1} = 1 + 2\,\Sigma_q \eta^q \cos qX, \quad q = 1, 2, \ldots \tag{7.2}$$

The constant term in the product is

$$2\,\Sigma_p\, \eta^{|i-p|}\, J_p(ie),$$

which is therefore the required coefficient of $\cos iw$. The constant term by (2.4), (7.2) is unity.

The expression (7.1) is df/dw, when f, w are the true and mean anomalies in an elliptic orbit. On integration we obtain for the expression of the true anomaly in terms of the mean,

$$f = w + \Sigma_i \frac{1}{i}\left\{\Sigma_p \eta^{|i-p|}\, J_p(ie)\right\}\sin iw,$$

where $p = 0, \pm 1, \pm 2, \ldots, i = 1, 2, \ldots$.

8. An immediate proof of the theorem that the second term of the disturbing function contributes nothing to its constant part is furnished by the last theorem of § 5.

In the usual notation,

$$R = \frac{m}{\Delta} - \frac{mrS}{r'^2}, \quad \Delta^2 = r^2 + r'^2 - 2rr'S,$$

$S = \cos^2 {}^1/_2 i \cos(f + \tilde{\omega} - f' - \tilde{\omega}') + \sin^2 {}^1/_2 i \cos(f + \tilde{\omega} + f' + \tilde{\omega}' - 2\theta)$. On the multiplication of R by $r^2r'^2$ and the substitution of $a(1 - e^2)\cdot(1 + e\cos f)^{-1}$ for r in r^3S we see at once that when r^3S is expressed as a sum of periodic terms, every term must contain f' in its angle and there can therefore be no constant term.

[1] *Proc. Nat. Acad. Sci.*, **16**, 1(1930).

CONTENT-PRESERVING TRANSFORMATIONS

By M. S. KNEBELMAN

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

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By a generalized metric space of n dimensions we shall understand one in which arc length is defined by a positively homogeneous function of the second degree in $dx^1, \ldots, dx^n$; $x^1, \ldots, x^n$ being the coördinates of any point. In a space of this sort the fundamental tensor $g_{ij}(x, dx)$ is homogeneous of degree zero in dx instead of being independent of dx, as it is in a Riemann space. For this reason all measurements in our space are relative to a chosen direction.

If $\lambda^i_{(\alpha)}(x, dx)$, $\alpha = 1, \ldots, k \leq n$, are k independent contravariant vectors, that is, if $\left\| \lambda^i_{(\alpha)} \right\|$ is of rank k, the content of the k-cell defined by them is given by

$$V^2 = g_{a_1 \ldots a_k;\, i_1 \ldots i_k} \prod_{\alpha=1}^{k} \lambda^{a_\alpha}_{(\alpha)} \lambda^{i_\alpha}_{(\alpha)},$$

where the summation convention is used only for repeated indices and where $g_{a_1 \ldots a_k;\, i_1 \ldots i_k}$ denote the k-rowed minors of the non-vanishing determinant $|g_{ij}|$, the rows and columns of the minor being indicated by the a's and i's, respectively.

Our problem is to find the conditions that the fundamental tensor must satisfy in order that there shall exist a finite continuous group of extended point transformations which shall carry every k-cell (for a given k) into a k-cell of the same content.

If $k = 1$ the transformation in question—if it exists—is a motion of the space into itself and in that case the content of every k-cell for every $k = 1, \ldots, n$ is preserved.

As in the problem of motion we shall consider the infinitesimal transformations of the group, since the general theory of finite continuous groups establishes the existence of the corresponding finite transformations. Let

$$\bar{x}^i = x^i + \xi^{i\,(x)}\delta u \tag{2}$$

be an infinitesimal transformation defined by the vector $\xi^{i\,(x)}$. Then the components of the fundamental tensor at $\bar{x}$ are given by

$$g_{ij}(\bar{x}, d\bar{x}) = g_{ij}(x, dx) + \left(\frac{\partial g_{ij}}{\partial x^k}\xi^k + \frac{\partial g_{ij}}{\partial dx^k}\frac{\partial \xi^k}{\partial x^m}dx^m\right)\delta u + \ldots \tag{3}$$

while the vector $\lambda^i(x,dx)$ is carried into the vector

$$\bar{\lambda}^i(\bar{x}, d\bar{x}) = \lambda^i + \frac{\partial \xi^i}{\partial xj}\lambda^j\delta u, \tag{4}$$

in accordance with the law of transformation of vectors.

It therefore follows that

$$\overline{V}^2 = \left[g_{a_1\ldots a_k;\, i_1\ldots i_k} + (g_{a;\, i,m}\,\xi^m + g_{a;i\,.\,m}\frac{\partial \xi^m}{\partial xj}dx^j)\delta u\right].$$

$$\cdot \prod_{\alpha=1}^{k} \lambda^{\sigma_\alpha}_{(\alpha)} \cdot \lambda^{\tau_\alpha}_{(\alpha)}(\delta^{a_\alpha}_{\sigma_\alpha} + \xi^{a_\alpha}_{,\sigma_\alpha}\delta u)(\delta^{i_\alpha}_{\tau_\alpha} + \xi^{i_\alpha}_{,\tau_\alpha}\delta u),$$

where a subscript preceded by a comma denotes partial differentiation with respect to x and that by a period, with respect to dx.

If a content-preserving transformation is to exist in the given space $\overline{V}^2 = V^2$ for all vectors, $\lambda^i_{(\alpha)}$ and the equations arising from making $\overline{V}^2$ identically equal to V^2 will be the differential equations that the vector ξ^i must satisfy in order to define an infinitesimal content-preserving transformation.

Neglecting power of δu higher than the first these conditions give

$$(g_{a;\, i,m}\xi^m + g_{a;\, i,m}\,\xi^m_{,p}\,dx^p)\prod_{\alpha=1}^{k}(\delta^{a_\alpha}_{\tau_\alpha}\delta^{i_\alpha}_{j_\alpha} + \delta^{a_\alpha}_{j_\alpha}\delta^{i_\alpha}_{\tau_\alpha})$$

$$+ g_{a;\, i}\sum_{\alpha=1}^{k}(\delta^{a_\alpha}_{\tau_\alpha}\xi^{i_\alpha}_{,j_\alpha} + \delta^{a_\alpha}_{j_\alpha}\xi^{i_\alpha}_{,\tau_\alpha} + \delta^{i_\alpha}_{\tau_\alpha}\xi^{a_\alpha}_{,j_\alpha} + \delta^{i_\alpha}_{j_\alpha}\xi^{a_\alpha}_{,\tau_\alpha})\cdot\prod_{\beta=1}^{k}{}'(\delta^{a_\beta}_{\tau_\beta}\delta^{i_\beta}_{j_\beta}$$

$$+ \delta^{a_\beta}_{j_\beta}\delta^{i_\beta}_{\tau_\beta}) = 0, \tag{5}$$

where $\prod'_{\beta=1}^{k}$ indicates the product of all factors except the αth one.

Because of symmetry these equations are reducible to

$$2(g_{\xi;j,m}\,\xi^m + g_{\xi;j,m}\,\xi^m_{,p}\,dx^p) + \sum_{\alpha=1}^{k} g_{\xi_1\ldots a\ldots\xi_k;\,j_1\ldots i\ldots j_k}(\delta^a_{\xi_\alpha}\,\xi^i_{,j_\alpha} + \delta^a_{j_\alpha}\,\xi^i_{,\xi_\alpha} + \delta^i_{\xi_\alpha}\,\xi^a_{,j_\alpha} + \delta^i_{j_\alpha}\,\xi^a_{,\xi_\alpha}) = 0. \quad (6)$$

Using the fact[1] that

$$g_{\xi_1\ldots\xi_k;\,j_1\ldots j_k} = \delta^{\gamma_1\ldots\gamma_k}_{\xi_1\ldots\xi_k} \prod_{\alpha=1}^{k} g_{\gamma_\alpha j_\alpha}$$

and multiplying (6) by $\prod_{\delta=2}^{k} g^{\xi\delta}g^{j\delta}$, contraction gives after considerable reduction,

$$2(g_{\gamma_1 j_1,m}\,\xi^m + g_{\gamma_1 j_1,m}\,\xi^m_{,p}\,dx^p)\delta^{\gamma_1\gamma_2\ldots\gamma_k}_{\xi_1\gamma_2\ldots\gamma_k} + 2\sum_{\alpha=2}^{k} \delta^{\gamma_1\gamma_2\ldots\gamma_\alpha\ldots\gamma_k}_{\xi_1\gamma_2\ldots\xi_\alpha\ldots\gamma_k} g_{\gamma_1 j_1}\,g^{\xi_\alpha j_\alpha}\,(g_{\gamma_\alpha j_\alpha,m}\,\xi^m + g_{\gamma_\alpha j_\alpha,m}\,\xi^m_{,p}\,dx^p) + g_{\gamma_1 i}\,\delta^{\gamma_1\gamma_2\ldots\gamma_k}_{a\gamma_2\ldots\gamma_k}\,(\delta^a_{\xi_1}\,\xi^i_{,j_1} + \delta^a_{j_1}\,\xi^i_{,\xi_1} + \delta^i_{\xi_1}\,\xi^a_{,j_1}) + \sum_{\alpha=2}^{k} \delta^{\gamma_1\gamma_2\ldots\gamma_\alpha\ldots\gamma_k}_{\xi_1\gamma_2\ldots a\ldots\gamma_k}\,g_{\gamma_1 j_1}\,g_{\gamma_\alpha i}g^{\xi_\alpha j_\alpha}\,(\delta^a_{\xi_\alpha}\,\xi^i_{,j_\alpha} + \delta^a_{j_\alpha}\,\xi^i_{,\xi_\alpha} + \delta^a_{\xi_\alpha}\xi^a_{,j_\alpha} + \delta^i_{j_\alpha}\xi^a_{,\xi_\alpha}) = 0.$$

Since[1] $\delta^{\gamma_1\gamma_2\ldots\gamma_k}_{\beta_1\gamma_2\ldots\gamma_k} = \dfrac{(n-1)!}{(n-k)!}\,\delta^{\gamma_1}_{\beta_1}$, these last equations become

$$(n-k)h_{ij} + (k-1)g_{ij}h = 0, \quad (7)$$

where

$$h_{ij} \equiv g_{ik}\frac{\partial\xi^k}{\partial x^j} + g_{kj}\frac{\partial\xi^k}{\partial x^i} + \xi^k\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ij}}{\partial dx^k}\frac{\partial\xi^k}{\partial x^m}\,dx^m \quad (8)$$

and

$$h \equiv g^{ij}h_{ij}.$$

It has been shown[2] that $h_{ij} = 0$ are the generalized equations of Killing, a solution of which—if it exists—defines a motion of the metric space into itself.

Multiplying (7) by g^{ij}, contracting the i's and j's we get

$$k(n-1)h = 0$$

and since $k \neq 0$ and $n \neq 1$, $h = 0$. Hence $(n-k)h_{ij} = 0$ and if $k < n$, $h_{ij} = 0$ and the desired transformation must be a motion. We therefore have

If a generalized metric space is such that there exists a transformation preserving the content of every k-cell for a given $k < n$, this transformation is a motion and thus will preserve the content of every k-cell for every value of k.

The only case that remains to be considered is $k = n$. The volume defined by n independent vectors is given by

$$V = \sqrt{g}\,|\,\lambda^i_{(\alpha)}\,|$$

and therefore

$$\overline{V} = \left[\sqrt{g} + \left(\frac{\partial\sqrt{g}}{\partial x^k}\,\xi^k + \frac{\partial\sqrt{g}}{\partial dx^k}\,\frac{\partial\xi^k}{\partial x^m}\,dx^m\right)\delta u\right]\left\{|\,\lambda^i_{(\alpha)}\,| + \frac{\partial\,|\,\lambda^i_{(\alpha)}\,|}{\partial\lambda^\alpha_{(\beta)}}\,\frac{\partial\xi^j}{\partial x^s}\,\lambda^s_{(\beta)}\delta_u\right\} = V + |\lambda^i_{(\alpha)}|.h.\delta_u.$$

Hence if volume is to be preserved we must have $h = 0$, since $|\,\lambda^i_{(\alpha)}\,|$ does not vanish. Or explicitly written, the necessary and sufficient conditions that ξ^i must satisfy in order to define a volume-preserving transformation is

$$\frac{\partial\xi^i}{\partial x^i} + \xi^i\,\frac{\partial\log\sqrt{g}}{\partial x^i} + \frac{\partial\log\sqrt{g}}{\partial dx^i}\,\frac{\partial\xi^i}{\partial xj}\,dxj = 0. \qquad (9)$$

The above equation must hold for all values of dx and is therefore equivalent to a system of partial differential equations which must admit a solution if the space is to admit a volume-preserving transformation. A solution of (9) may exist which does not satisfy $h_{ij} = 0$ and therefore a volume-preserving transformation need not preserve the content of cells of lower dimensionality. In fact, if the space be Riemannian, or more generally if the space be such that g is a point function, a volume-preserving transformation always exists; for in this case (9) becomes $\partial(\xi^i\sqrt{g})/\partial x^i = 0$. Therefore a vector density with vanishing divergence defines a volume-preserving transformation.

NOTE IN PROOF: In an entirely similar manner it can be shown that every transformation which carries every k-cell, $k < n$, into a proportional k-cell also carries every vector into one of proportional length. That is, if a transformation is such that $\overline{V}_k^2 = V_k^2 + kf(x)V_k^2\delta u$, then $\overline{L}^2 = L^2 + f(x)L^2\delta u$, L being the length of any vector.

[1] For the theory of determinants and generalized Kronecker deltas used in this paper see CAMBRIDGE TRACT No. 24 by Oswald Veblen.

[2] Cf. M. S. Knebelman, *Amer. J. Math.*, **51**, 1929 (527–564).

SIMPLICITY WITH RESPECT TO CERTAIN QUADRATIC FORMS

BY E. T. BELL

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

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1. *Simplicity.*—Let n denote an arbitrary integer >0, and m an odd interger >0. Write $N[n = f]$ for the number of representations of n in the quadratic form f when this number is finite. If $N[n = f]$ is a polynomial $P(n)$ in the real divisors of n alone, $N[n = f]$ is, by definition, *simple;* similarly for $N[m = f]$. In previous papers,[1,2,3] I have considered simplicity when f is a sum of an even number of squares. For an odd number >1 of squares, no $N[n = f]$ is simple, although if n be restricted to be a square, say k^2, then $N[k^2 = f]$ is simple when f is a sum of 3, 5 or 7 squares, and for one case (k restricted) of 11 squares.

To discuss simplicity for forms f in an odd number of variables, the above definition was extended in a former paper[4] as follows. The extension leads to theorems that are actually useful in calculations of numbers of representations, and it was devised with this end in view. Let Σ refer to all positive, zero or negative integers t which are congruent modulo M to a fixed integer T, and which render the arguments of the summand >0. Let $P(n)$ denote a polynomial in the real divisors alone of n, and let c be a constant integer >0. Then if $N[n = f]$ is of the form $A\Sigma P(n - ct^2)$, where A does not depend upon n, it is called *simple.*

In the previous paper,[4] an exhaustive discussion of simplicity was given for sums of 3, 5, 7, 9, 11 and 13 squares; in the present note, we state general theorems for simplicity with respect to pure quadratic forms $S_{2r} + 2^{a+2}S_1$, where S_j denotes a sum of j squares, and r, a are arbitrary constant integers >0. These theorems are the first of their kind, and are the first instances of the like for forms $S_{2r} + 2^{a+2}S_{2h-1}$, where r, h, a are arbitary constant integers >0. As the proofs are short, they are sufficiently indicated in §3.

2. *Theorems.*—The summations refer to all positive, zero or negative integers t such that the arguments of the summands are >0; S_j is as in §1, and a is an arbitrary constant integer ≥ 0.

THEOREM 1. *If $A(r)$ is a function of r alone, and $m \equiv 1 \bmod 4$, and if*

$$N[m = S_{4r} + 2^{a+2}S_1] = A(r)\Sigma\zeta_s(m - 2^{a+2}t^2), \qquad (1)$$

where $\zeta_s(n)$ denotes the sum of the sth powers of all the real positive divisors of n, then, necessarily,

$$s = 2r - 1, A(r) = 8r.$$

The only values of r for which (1) *holds are $r = 1, 2$.*

THEOREM 2. *If $A(r)$ is a function of r alone, and $m \equiv 3 \bmod 4$, and if*

$$N[m = S_{4r} + 2^{a+2}S_1] = A(r)\Sigma\,\zeta_s(m - 2^{a+2}t^2), \qquad (2)$$

then, necessarily,

$$s = 2r - 1,\ A(r) = \frac{32r(2r-1)(4r-1)}{3(3^{2r-1}+1)}.$$

The only values of r for which (2) holds are $r = 1, 2$.

THEOREM 3. *If $A(r)$ is a function of r alone, and $m \equiv 1 \bmod 4$, and if*

$$N[m = S_{4r+2} + 2^{a+2}S_1] = A(r)\Sigma\xi_s(m - 2^{a+2}\,t^2), \qquad (3)$$

where $\xi_s(n)$ denotes the sum of the sth powers of the real positive divisors of the form $4h + 1$ of n minus the like sum for the divisors of the form $4h + 3$, then, necessarily,

$$s = 2r,\quad A(r) = 8r + 4.$$

The only values of r for which (3) holds are $r = 0, 1$.

THEOREM 4. *If $A(r)$ is a function of r alone, and $m \equiv 3 \bmod 4$, and if*

$$N\,[m = S_{4r+2} + 2^{a+2}\,S_1] = A(r)\Sigma\xi_s(m - 2^{a+2}\,t^2), \qquad (4)$$

then, necessarily,

$$s = 2r,\ A(r) = -\frac{32r(2r+1)(4r+1).}{3(3^{2r}-1)}$$

The only values for which (4) holds are $r = 0, 1, 2$.

THEOREM 5. *If $A(r)$ is a function of r alone, and m is odd, and if*

$$N[2m = S_{4r+4} + 2^{a+2}\,S_1] = A(r)\Sigma\zeta_s(m - 2^{a+2}\,t^2), \qquad (5)$$

then, necessarily,

$$s = 2r + 1,\ A(r) = 8\,(r+1)(4r+3).$$

The only values of r for which (4) holds are $r = 0, 1, 2$.

It is to be noticed that Theorems 1–5 hold for each particular value of a. The next differs from the preceding in that the integer n represented is arbitrary. It does not include 1–5, and is independent of them.

THEOREM 6. *If $A(r)$ is a function of r alone, and n is an arbitrary integer >0, and if*

$$N[n = S_{2r} + bS_1] = A(r)\,\Sigma\,\phi\,(an - bt^2),$$

where a, b are constant integers >0, and ϕ is simple (as first defined), then, necessarily $r = 1, 2, 3$ or 4.

3. *Proofs.*—All follow immediately from the results for S_{4r}, S_{4r+2}, S_{4r+4}, S_{2r} in the papers[1,2,3] cited, on an obvious application of the next.

LEMMA 1. *If* $n \equiv j \bmod 4$, *so that* $n - 2^{a+2}t^2 \equiv j \bmod 4$, *and if*

$$N[n = S_j + 2^{a+2}t^2] = A\Sigma\psi(n - 2^{a+2}t^2), \tag{6}$$

where A *is independent of* n *and* ψ *is simple as first defined, then* (6) *implies*

$$N[n = S_j] = A\psi(n), \tag{7}$$

and (7) *implies* (6).

That (7) implies (6) is obvious. To see that (6) implies (7), equate the generating functions of the functions on the left and right of (6), to get an identity in the parameter q of the generating function. Divide both sides of this identity by

$$\Sigma_q^{2^{a+2}t^2}(t = 0, \pm 1, \pm 2, \ldots),$$

and thus get the identity which implies (7). Clearly the lemma holds also for ψ not simple.

LEMMA 2. *If* n *is an arbitrary integer* >0, *and* A *is independent of* n, *each of the following implies the other,*

$$N[n = S_j + t^2] = A\Sigma\psi(n - t^2), \; N[n = S_j] = A\psi(n).$$

[1] E. T. Bell, *Bull. Am. Math. Soc.*, **35**, 695 (1929).
[2] E. T. Bell, *J. London Math. Soc.*, **4**, 279 (1929).
[3] E. T. Bell, *J. für die r.u.a. Math.*, to appear shortly.
[4] E. T. Bell, *Amer. J. Math.*, **42**, 168 (1920).

THE DIFFERENTIAL GEOMETRY OF A CONTINUOUS INFINITUDE OF CONTRAVARIANT FUNCTIONAL VECTORS[1]

BY ARISTOTLE D. MICHAL

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

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1. *Introduction.*—A general theory of function space affinely connected manifolds has been developed by the author in several publications.[2] In this paper I propose to give a number of new results pertaining to the differential geometry and invariant theory of a continuous infinitude of contravariant functional vectors. An application is made of these results to the differential geometry of functional group vectors of infinite groups of functional transformations. It is my intention to publish the complete results and proofs elsewhere.

2. *Infinitude of Contravariant Functional Vectors.*—Let $\eta_i[y]$ be a covariant functional vector, then the pair of functionals $\xi^\alpha[y]$, $\xi_i^\alpha[y]$ will

be said to constitute an infinitude of contravariant functional vectors if

$$L_i[\eta] = \xi^{(i)}\eta_i + \xi_i^\alpha \eta_\alpha$$

is an absolute functional form in $\eta_i[y]$. Throughout the paper we shall assume that $\xi^i[y] \neq 0$ and that there does not exist a functional $\lambda^i[y]$, other than $\lambda^i \equiv 0$, that satisfies the integral equation (linear independence)

$$\lambda^i[y]\xi^i[y] + \lambda^\alpha[y]\,\xi_\alpha^i[y] = 0.$$

In other words we shall assume that the Fredholm determinant

$$D[\xi_\mu^\lambda/\xi^{(\mu)}] \neq 0.$$

THEOREM I. *A necessary and sufficient condition that the pair of functionals $\xi^\alpha[y]$, $\xi_i^\alpha[y]$ form an infinitude of contravariant functional vectors is that they transform in accordance with the law*

$$\left.\begin{aligned} \bar{\xi}^i[\bar{y}] &= \xi^i[y]\,{}_{,i}\bar{y}^{(i)} \\ \bar{\xi}_i^\alpha[\bar{y}] &= \xi_i^\alpha[y]\,{}_{,\alpha}\bar{y}^{(\alpha)} + \xi_i^\lambda\,\bar{y}_{,\lambda}^\alpha + \xi^{(i)}\,\bar{y}_{,i}^\alpha \end{aligned}\right\} \tag{1}$$

under the functional transformations

$$\bar{y}^i = f^i[y].$$

It is to be observed that the inversion of the law of transformation (1) involves the solution of a system of mixed linear *integral equations.*

THEOREM II. *The Fredholm determinant*

$$D[\xi_\sigma^\rho/\xi^{(\sigma)}]$$

is a relative scalar functional invariant of weight minus one of the infinitude of contravariant functional vectors $\xi_\sigma^\rho[y]$, $\xi^\rho[y]$.

Put

$$\eta^i[y] = 1/\xi^i[y],\ \eta_\alpha^i[y] = -\frac{D_\alpha^i[\xi_\mu^\lambda/\xi^{(\mu)}]}{\xi^i D[\xi_\tau^\sigma/\xi^{(\tau)}]}.$$

3. *Asymmetric Functional Affine Connection.*

THEOREM III. *The functionals $\xi^i[y]$, $\xi_\alpha^i[y]$ determine the asymmetric functional affine connection*

$$\left.\begin{aligned} L_{\alpha\beta}^i[y] &= -\xi_{,\beta}^i\eta_\alpha^i - \xi_{\alpha,\beta}^i\,\eta^{(\alpha)} - \xi_{\sigma,\beta}^i\,\eta_\alpha^\sigma - {}_{,\beta}\xi^i\,\eta_\alpha^{(\beta)} \\ M_\alpha^i[y] &= -{}_{,\alpha}\xi_\alpha^i\,\eta^{(\alpha)} \\ N_\alpha^i[y] &= -{}_{,i}\xi^{(i)}\,\eta_\alpha^i - {}_{,i}\xi_\alpha^{(i)}\,\eta^{(\alpha)} - {}_{,i}\xi_\sigma^{(i)}\,\eta_\alpha^\sigma \\ O_\alpha^i[y] &= -\xi_{,\alpha}^i\,\eta^i \\ P^i[y] &= -{}_{,i}\xi^i\,\eta^i. \end{aligned}\right\} \tag{2}$$

Let

$$\left.\begin{aligned} \omega^i &= P^i\,\delta y^i + O_\alpha^i\,\delta y^\alpha \\ \omega_\sigma^i &= N_\sigma^i\,\delta y^i + M_\sigma^i\,\delta y^{(\sigma)} + L_{\sigma\alpha}^i\,\delta y^\alpha, \end{aligned}\right\} \tag{3}$$

where the functionals L, M, N, O, P constitute an arbitrary asymmetric connection, not necessarily the one defined by (2). In terms of the Pfaffian functional forms ω^i and ω^i_σ, the covariant functional differential of ξ^i, ξ^i_α is given by

$$\left.\begin{aligned} D\xi^k[y] &= \delta\xi^k[y] + \xi^k\,\omega^k \\ D\xi^i_k[y] &= \delta\xi^i_k[y] + \xi^i_k\,\omega^i + \xi^\sigma_k\,\omega^i_\sigma + \xi^{(k)}\,\omega^i_k \end{aligned}\right\} \tag{4}$$

THEOREM IV. *The covariant functional derivative of the infinitude of functional vectors ξ^i, ξ^i_α, based on an arbitrary affine connection is given by the sequence of functionals ${}_k;\xi^{(k)}$, $\xi^k_{;l}$, ${}^i;\xi^i_k$, ${}_k;\xi^i_k$, $\xi^i_{k;l}$ where*

$$\left.\begin{aligned} {}_k;\xi^{(k)} &= {}_k,\xi^{(k)} + \xi^k\,P^k \\ \xi^k_{;l} &= \xi^k_l + \xi^k\,O^k_l \\ {}^i;\xi^i_k &= {}^i,\xi^i_k + \xi^i_k P^i + \xi^\sigma_k\,N^i_\sigma + \xi^{(k)}\,N^i_k \\ {}_k;\xi^i_k &= {}_k,\xi^i_k + \xi^{(k)}\,M^i_k \\ \xi^i_{k;l} &= \xi^i_{k,l} + \xi^i_k\,O^i_l + \xi^{(l)}_k\,M^i_l + \xi^\sigma_k\,L^i_{\sigma l} + \xi^{(k)}L^i_{kl}. \end{aligned}\right\} \tag{5}$$

THEOREM V. *The covariant functional derivative of ξ^i, ξ^i_α based on the functional affine connection* (2) *vanishes*:

$${}_k;\xi^{(k)} = 0,\ \xi^k_{;l}\ 0, = {}^i;\xi^i_k = 0,\ {}_k;\xi^i_k = ,\ \xi^i_{k;l} = 0.$$

As a consequence of this theorem, the result is obtained that a sequence of five functionals (the functional curvature tensor of the asymmetric connection) vanishes.

4. *Functional Group Vectors.*—From the asymmetric functional affine connection (2), a symmetric one is defined by the four functionals

$$\left.\begin{aligned} \mathfrak{L}^i_{\alpha\beta} &= {}^1/_2\,(L^i_{\alpha\beta} + L^i_{\beta\alpha}) \\ \mathfrak{M}^i_\alpha &= M^i_\alpha \\ \mathfrak{N}^i_\alpha &= {}^1/_2\,(N^i_\alpha + O^i_\alpha) \\ \mathfrak{P}^i &= P^i \end{aligned}\right\} \tag{6}$$

Let the five functionals of the curvature tensor based on (6) be denoted by $B^i_{\alpha\beta\gamma}$, $C^i_{\alpha\beta}$, $D^i_{\alpha\beta}$, E^i_α, F^i_α.

If we now *specialize our functional vectors* and consider $\xi^i[y]$, $\xi^i_\alpha[y]$ to be the functional group vectors of the first parameter group of a functional group with one arbitrary function,[3] the following theorem holds good.

THEOREM VI. *The nineteen functionals that make up the covariant functional derivative of the curvature tensor $B^i_{\alpha\beta\gamma}$, $C^i_{\alpha\beta}$, $D^i_{\alpha\beta}$, E^i_α, F^i_α all vanish.*

[1] Presented to the Amer. Math. Society, Des Moines meeting, December, 1929.

[2] A. D. Michal, *Amer. J. Math.*, **50**, 473–517 (1928) (presented to the Amer. Math. Soc. at the New York meeting, October, 1927—cf. *Bull. Amer. Math. Soc.*, **34**, 8–9 (1928)); these PROCEEDINGS, **16**, 88–94 (1930) (presented to the Amer. Math. Soc. at the New York meeting, March, 1929—cf. *Bull. Amer. Math. Soc.*, **35**, 438–439 (1929)).

[3] G. C. Moisil, *Comptes Rendus*, **188**, 691–692 (1929).

PROJECTIVE FUNCTIONAL TENSORS AND OTHER ALLIED FUNCTIONALS

BY ARISTOTLE D. MICHAL

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

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1. *Introduction.*—Some time ago I discovered a sequence of functionals[1] that is the correspondent in function space[2] of Weyl's projective curvature tensor in n dimensions.[3] Since then, I have succeeded in finding a sequence of non-tensor functionals[4] with the property that each functional of the sequence remains unaltered under an arbitrary projective functional transformation[1] of the functional affine connection. It is the object of this note to outline briefly the results obtained in the above studies. A detailed presentation with proofs as well as an account of numerous results and theorems that flow out of the ideas and methods of the present note is reserved for a series of papers to be published elsewhere.

2. *Projective Functional Transformations of Functional Affine Connections.*—Let $\Gamma^i_{\alpha\beta}[y]$, $M^i_\alpha[y]$, $N^i_\alpha[y]$, $P^i[y]$ be the functionals of an arbitrary symmetric functional affine connection.[2] The most general projective functional transformation of such a functional affine connection is given by

$$\left.\begin{aligned} {}'\Gamma^i_{\alpha\beta}[y] &= \Gamma^i_{\alpha\beta}[y],\ {}'M^i_\alpha[y] = M^i_\alpha[y] \\ {}'N^i_\alpha[y] &= N^i_\alpha[y] + \varphi_\alpha[y],\ {}'P^i[y] = P^i[y], \end{aligned}\right\} \tag{1}$$

where $\varphi_\alpha[y]$ is an arbitrary covariant functional vector. More generally, the projective functional transformation of an asymmetric functional affine connection is determined by

$$\left.\begin{aligned} {}'\Gamma^i_{\alpha\beta}[y] &= \Gamma^i_{\alpha\beta}[y],\ {}'M^i_\alpha[y] = M^i_\alpha[y],\ {}'S^i_\alpha[y] = S^i_\alpha[y] + \varphi_\alpha[y], \\ {}'\Omega^i_{\alpha\beta} &= \Omega^i_{\alpha\beta},\ {}'T^i_{\alpha\beta}[y] = T^i[y] + \varphi_\alpha[y],\ {}'P^i[y] = P^i[y], \end{aligned}\right\} \tag{2}$$

where $\varphi_\alpha[y]$ is again an arbitrary covariant functional vector and

$$\left.\begin{aligned} \Gamma^i_{\alpha\beta} &= {}^1/_2(L^i_{\alpha\beta} + L^i_{\beta\alpha}),\ S^i_\alpha = {}^1/_2(N^i_\alpha + O^i_\alpha) \\ \Omega^i_{\alpha\beta} &= {}^1/_2(L^i_{\alpha\beta} - L^i_{\beta\alpha}),\ T^i_\alpha = {}^1/_2(N^i_\alpha - O^i_\alpha). \end{aligned}\right\} \tag{3}$$

The functionals $\Omega^i_{\alpha\beta}[y]$ and $T^i_\alpha[y]$ are the two distinct functionals of the *torsion functional tensor.* Incidentally in this connection we state the theorem that

$$\Omega^i_{jk}\,\xi^j\,\eta^k u_i + T^i_k u_i\,(\xi^i\eta^k - \xi^k\eta^i) \tag{4}$$

is an absolute cubic functional form in the contravariant functional vectors $\xi^i[y]$, $\eta^i[y]$ and in the covariant functional vector $u_i[y]$.

3. *The Analogue of Weyl's Tensor in Function Space.*—Consider now a symmetric functional affine connection.[2] Let

$$B^i_{\alpha\beta\gamma}[y],\ C^i_{\alpha\beta}[y],\ D^i_{\alpha\beta}[y],\ E^i_\alpha[y],\ F^i_\alpha[y]$$

stand for the five distinct functionals of the *functional curvature tensor.*[2] Define now a sequence of five other functionals

$$\mathfrak{B}^i_{\alpha\beta\gamma}[y],\ \mathfrak{C}^i_{\alpha\beta}[y],\ \mathfrak{D}^i_{\alpha\beta}[y],\ \mathfrak{E}^i_\alpha[y],\ \mathfrak{F}^i_\alpha[y]$$

by

$$\left.\begin{aligned}
\mathfrak{B}^i_{\alpha\beta\gamma}[y] &= B^i_{\alpha\beta\gamma}[y] \\
\mathfrak{C}^i_{\alpha\beta}[y] &= C^i_{\alpha\beta}[y] \\
\mathfrak{D}^i_{\alpha\beta}[y] &= D^i_{\alpha\beta}[y] - \frac{1}{b-a}\int_a^b D^\sigma_{\alpha\beta}[y]d\sigma \\
E^i_\alpha[y] &= E^i_\alpha[y] - \frac{1}{b-a}\int_a^b E^\sigma_\alpha[y]d\sigma \\
F^i_\alpha[y] &= F^i_\alpha[y].
\end{aligned}\right\} \tag{5}$$

This new sequence of five functionals can be considered as the five distinct functional coefficients of a quartic functional form[2] $Q[\xi, \eta, \zeta, u]$ in the contravariant functional vectors $\xi^i[y]$, $\eta^i[y]$, $\zeta^i[y]$ and covariant functional vector $u_i[y]$, with symmetry conditions

$$\left.\begin{aligned}
&Q[\xi, \eta, \zeta, u] = -Q[\xi, \zeta, \eta, u] \\
&Q[\xi, \eta, \zeta, u] + Q[\zeta, \xi, \eta, u] + Q[\eta, \zeta, \xi, u] = 0.
\end{aligned}\right\} \tag{6}$$

Consequently the functionals $\mathfrak{B}^i_{\alpha\beta\gamma}$, $\mathfrak{C}^i_{\alpha\beta}$, $\mathfrak{D}^i_{\alpha\beta}$, $\mathfrak{E}^i_\alpha$, $\mathfrak{F}^i_\alpha$ are the functionals of a functional tensor of rank four, covariant of rank three and contravariant of rank one, with symmetry conditions that are easily obtained from (6). But, what is more important for the purpose of this paper is that this same sequence of functionals possesses the remarkable property that each functional of the sequence is an invariant functional of the four functionals $\Gamma^i_{\alpha\beta}$, M^i_α, N^i_α, P^i and their first functional derivatives under the infinite group of projective functional transformations (1). The set of functionals $\mathfrak{B}^i_{\alpha\beta\gamma}, \ldots, \mathfrak{F}^i_\alpha$ will be said to constitute the *projective curvature tensor in function space.*

4. *Projective Functional Connections.*—From an asymmetric[2] functional connection $L^i_{\alpha\beta}$, M^i_α, N^i_α, O^i_α, P^i it is possible to form the sequence of functionals $\lambda^i_{\alpha\beta}$, μ^i_α, ν^i_α, o^i_α, π^i defined by

$$\left.\begin{aligned}
&\lambda^i_{\alpha\beta} = L^i_{\alpha\beta},\ \mu^i_\alpha = M^i_\alpha,\ \nu^i_\alpha = N^i_\alpha - \frac{1}{b-a}\int_a^b N^\sigma_\alpha d\sigma, \\
&o^i_\alpha = O^i_\alpha,\ \pi^i = P^i.
\end{aligned}\right\} \tag{7}$$

Each functional of this sequence is invariant under the infinite group determined by (2).

In particular, if we specialize our work and consider a symmetric functional connection $\Gamma^i_{\alpha\beta}$, M^i_α, N^i_α, P^i, each member of the sequence of functionals $\gamma^i_{\alpha\beta}$, μ^i_α, ν^i_α, π^i defined by

$$\gamma^i_{\alpha\beta} = \Gamma^i_{\alpha\beta},\ \mu^i_\alpha = M^i_\alpha,\ \nu^i_\alpha = N^i_\alpha - \frac{1}{b-a}\int_a^b N^\sigma_\alpha d\sigma,\ \pi^i = P^i \quad (8)$$

will have the property of remaining unaltered under the group (1). Because of this property, the four functionals defined by (8) will be said to constitute a *projective functional connection*[5] (symmetric—as distinguished from the asymmetric case (7)). I find that the law of transformation of the projective functional connection is given by

$$\left.\begin{aligned}
&{}_{i,}y^{(i)}\,\bar\gamma^i_{jk} + y^i_{,\alpha}\,\bar\gamma^\alpha_{jk} = y^i_{,jk} - y^i_{,j}\,{}_{j,}y^i_{,k}\,(1/{}_{j,}y^{(j)}) - y^i_{,k}\,{}_{k,}y^{(k)}_{,j}\,(1/{}_{k,}y^{(k)})\\
&+ \gamma^i_{jk}\,{}_{j,}y^{(j)}\,{}_{k,}y^{(k)} + \gamma^i_{j\sigma}\,{}_{j,}y^{(j)}\,y^\sigma_{,k} + \gamma^i_{\sigma k}\,y^\sigma_{,j}\,{}_{k,}y^{(k)} + \gamma^i_{\sigma\tau}y^\sigma_{,j}\,y^\tau_{,k}\\
&+ \mu^i_j\,{}_{j,}y^{(j)}\,y^j_{,k} + \mu^i_k\,y^{(k)}_{,j}\,{}_{k,}y^k + \mu^i_\sigma\,y^\sigma_{,j}\,y^\sigma_{,k} + (\nu^i_k - \nu^{(j)}_k)\,y^i_{,j}\,{}_{k,}y^{(k)}\\
&+ (\nu^i_\sigma - \nu^{(j)}_\sigma)\,y^i_{,j}\,y^\sigma_{,k} + (\nu^i_j - \nu^{(k)}_j)\,{}_{j,}y^{(j)}\,y^i_{,k} + (\nu^i_\sigma - \nu^{(k)}_\sigma)\,y^\sigma_{,j}\,y^i_{,k}\\
&+ \pi^i\,y^i_{,j}\,y^i_{,k} - \pi^{(j)}\,y^i_{,j}\,y^j_{,k} - \pi^{(k)}\,y^i_{,k}\,y^k_{,j}\\
&{}_{i,}y^{(i)}\,\bar\mu^i_j + y^i_{,\alpha}\,\bar\mu^\alpha_j = {}_{j,}y^i_{,j} - y^i_{,j}\,{}_{jj,}y^{(j)}\,(1/{}_{j,}y^j) - y^i_{,j}\,\pi^{(j)}\,{}_{j,}y^j\\
&+ \mu^i_j\,({}_{j,}y^{(j)})^2\\
&{}_{i,}y^{(i)}\,\bar\nu^i_j = {}_{i,}y^i_{,j} - \frac{{}_{i,}y^{(i)}}{b-a}\int_a^b (1/{}_{\sigma,}y^\sigma)\,{}^\sigma{}_{,}y^\sigma_j\,d\sigma\\
&+ \nu^i_j\,{}_{i,}y^{(i)}\,{}_{j,}y^{(j)} + \nu^i_\sigma\,{}_{i,}y^{(i)}\,y^\sigma_{,j}\\
&+ \pi^i\,{}_{i,}y^{(i)}\,y^i_{,j} - \frac{{}_{i,}y^{(i)}}{b-a}\int_a^b \pi^\sigma\,y_{,j}{}^\sigma\,d\sigma\\
&{}_{i,}y^i\,\bar\pi^i = {}_{ii,}y^{(i)} + \pi^i\,({}_{i,}y^{(i)})^2
\end{aligned}\right\} \quad (9)$$

5. *Concluding Remarks.*—If in the projective transformation of a given symmetric functional connection $\Gamma^i_{\alpha\beta}$, M^i_α, N^i_α, P^i, we choose the functional vector $\varphi_\alpha[y]$ as

$$\varphi_\alpha[y] = -\frac{1}{b-a}\int_a^b N^\sigma_\alpha\,d\sigma, \quad (10)$$

we obtain a unique *normal functional connection*,[5] $'\Gamma$, $'M$, $'N$, $'P$, for an arbitrarily given functional coördinate system y^i, with the property

$$\int_a^b {}'N^\sigma_\alpha\,d\sigma = 0. \quad (11)$$

For a *projective functional parameter*[5] p along a functional geodesic

$$p = k\left\{\int e^{-\frac{2}{b-a}\int\left[\int_a^b N^\sigma_\alpha \frac{\partial y^{(u)}}{\partial u}\,d\sigma\right]du}\right\}ds, \quad (12)$$

we have the theorem that a projective functional parameter p determined by y^i is an affine functional parameter for the normal functional connection determined by y^i.

Finally, a large number of theorems have been proved by me in connection with a theory of projective normal coördinates in function space and a theory of hybrid projective functional tensors under algebro-functional groups of the type

$$\left.\begin{aligned} x &= \bar{x} + F[y^i] \\ y^i &= f^i\,[\bar{y}^a]. \end{aligned}\right\}$$

[1] Michal, A. D., *Bull. Amer. Math. Soc.*, **35** (1929), 438–439.

[2] Michal, A. D., *Amer. J. Math.*, **50** (1928), 473–517; *Bull. Amer. Math. Soc.*, **34** (1928), 8–9; these PROCEEDINGS, **16** (1930), 88–94; *Ibid.*, **16** (1930), 162–164.

[3] Weyl, H., *Gottinger Nachrichten*, **1921**, pp. 99–112.

[4] Michal, A. D., *Des Moines Meeting of Amer. Math. Soc.*, December, 1929.

[5] Although our algebra and analysis is from the very nature of our subject different from those existent in differential geometries in a finite number of dimensions, I have emphasized as much as possible the many analogies that exist by a suitable choice of notation and terminology. A point in question is the projective connection and projective parameter of Tracy Yerkes Thomas, whose brilliant work on a projective theory of n-dimensional affinely connected manifolds (*Math. Zeitschrift*, **25** (1926), 723–733) together with the notable work of O. Veblen (these PROCEEDINGS, **14** (1928), pp. 154–166) and many others on the same subject has opened up beautiful vistas for the mathematician. For a masterly account of some of these n-dimensional researches as well as for references to the important work of Weyl, Cartan, and Schouten, the reader is referred to L. P. Eisenhart's colloquium lectures on "Non-Riemannian Geometry," Amer. Math. Soc., New York, 1927.

NON-ABELIAN GROUPS ADMITTING MORE THAN HALF INVERSE CORRESPONDENCIES

BY G. A. MILLER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS

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Let G represent any non-abelian group of order g which admits an automorphism under which more than half the operators correspond to their inverses. All such automorphisms must be of order 2 since in the square thereof more than half of the operators of G correspond to themselves and hence all of these operators have then this property. Let H represent one of the largest subgroups of G which satisfy the condition that each of their operators corresponds to its inverse under the given automorphism of G. It is known that H is abelian and that if s represents any operator of G which corresponds to its inverse under this automorphism but is not found in H then the number of operators which correspond to their inverses in the co-set of G with respect to H which involves s is exactly equal to the number of operators in H which are commutative with s. Moreover, every co-set of G with respect to H must involve at least one operator which corresponds to its inverse in this automorphism.

The special case when H is of index 2 is comparatively simple. In this case the number of operators which correspond to their inverses

under the given automorphism is equal to $g/2 + k$, where k represents the order of the subgroup composed of all the operators of H which are commutative with s. We proceed to find the possible values of k when g is given but when G is any one of the possible non-abelian groups of this order which satisfy the given condition. It is clear that without loss of generality we may assume that the order of s is 2 when H is of index 2, since s then transforms the operators of H according to an automorphism of order 2. When the order h of H is odd then k may be any divisor of H, including the identity but excluding h, while k must be even whenever h is even since s must then be commutative with an operator of order 2 contained in H. As k may then have any such even value less than h there results the following theorem: *It is possible to construct at least one non-abelian group of an arbitrary even order $g > 4$ such that it admits an automorphism in which all the operators of a subgroup of index 2 correspond to their inverses and exactly k additional operators correspond also to their inverses, where k is an arbitrary divisor of $g/2$ or an arbitrary even divisor of $g/2$ as this number is odd or even.* These divisors include unity but they exclude $g/2$, and k can have no other value than those given by this theorem.

It is not difficult to prove that g must be an even number whenever the non-abelian group G admits an automorphism in which more than half of its operators correspond to their inverses. For if g were odd the order of s would also be odd and s could not be commutative with more than one-third of the operators of H. Moreover, H could not then be invariant under G since s corresponds to its inverse under the given automorphism and the resulting commutators would also correspond to their inverses under this automorphism. If s were commutative with less than one-third of the operators of H the number of operators which would correspond to their inverses in the automorphism in question would clearly be less than $g/2$. Hence it remains only to consider the case when H is non-invariant under G and all the operators which correspond to their inverses in this automorphism and are not found in H are of order 3. If $g = 3h$ then the operators of H which are commutative with s constitute the central of G and the central quotient of G is of order 9. This is impossible since H is not invariant under G. It therefore results that $g > 3h$ and hence there are more than three co-sets with respect to H in G. That is, *there is no non-abelian group of odd order which admits an automorphism in which as many as half of its operators correspond to their inverses.* In view of these theorems it will be assumed in what follows that g is even and that h is less than $g/2$. We shall first prove that every operator of G which corresponds to its inverse in the given automorphism has its square in H. If s^2 were not found in H then s could not be commutative with more than one-third of the operators of H since H is a maximal sub-

group among those which have the property that each of their operators corresponds to its inverse in the given automorphism. If s were commutative with less than one-third of the operators of H it would not be possible for more than half of the operators to correspond to their inverses in this automorphism. Hence it remains only to consider the case when s is commutative with exactly one-third of the operators of H, and hence H involves a subgroup of index 3 composed of its operators which are commutative with s. Moreover, s^3 must appear in H.

If H were invariant this would imply that s would transform H according to an automorphism of order 3 since an automorphism of an abelian group in which exactly one-third of the operators correspond to themselves is either of order 2 or of order 3, and in the present case it could not be of order 2 since it is assumed that s^2 is not found in H. The assumption that the automorphism is of order 3 leads to a contradiction since s must correspond to its inverse in the given automorphism. It therefore results that H is a non-invariant subgroup of G if it does not include s^2. This is, however, again untenable since s and H would generate a group whose central would be of index 3 under H and the quotient group with respect to this central would involve at least four conjugate subgroups of order 3. This would imply that less than one-half of the operators of G would correspond to their inverses in the given automorphism and hence we have established the following theorem: *If more than half the operators of a non-abelian group correspond to their inverses in an automorphism of this group and if H represents any one of the largest subgroups composed of operators which correspond to their inverses under this automorphism then H must involve the square of every operator which corresponds to its inverse thereunder.*

From this theorem it results that all the operators of odd order in G which correspond to their inverses under the given automorphism must appear in H. It also results therefrom that every operator of G which corresponds to its inverse in this automorphism must transform H into a group in which all the operators correspond to their inverses and hence the complete set of conjugate subgroups to which H belongs under G is composed of subgroups in which every operator corresponds to its inverse. If H is a non-invariant subgroup under the operator s this operator could not be commutative with exactly one-third of the operators of H since otherwise it would be commutative with all the operators of a subgroup of index 3 contained in H. This is impossible since it must transform all the operators of odd order contained in H into operators of H and some of these operators could not appear in the given subgroup of index 3. If the operators of H which are commutative with s would constitute a subgroup of a larger index than 3 under H exactly half the operators of every co-set of G with respect to H, except the one involving s, would

correspond to their inverses in the given automorphism. This is impossible since in the central quotient group of the group generated by s and H the subgroup corresponding to H would be non-invariant and all of its operators could therefore not transform into themselves the operators of this quotient group corresponding to the co-set involving s. Hence it has been proved that all the operators in the co-sets of G with respect to H which do not transform H into itself are commutative with exactly half of the operators of H.

When H is invariant under G it is possible to construct G so that the index of H under G exceeds any given number. To prove this fact we may consider the case when H is composed of operators of order 2 besides the identity and when the commutator subgroup of G is of order 2. If the order of H is 2^m, $m > 1$, it is possible to extend H by an operator s of order 2 which is commutative with exactly half of its operators and thus to obtain a group of order 2^{m+1} which has exactly three abelian subgroups of order 2^m, two of which are composed of operators of order 2 besides the identity. When $m > 2$, we may extend this group of order 2^{m+1} by an operator s which is commutative with s and with the operators of a different subgroup of index 2 under H which includes the commutator subgroup of order 2. We thus obtain a group whose central is of order 2^{m-2} which involves abelian subgroups of order 2^m, but no larger abelian subgroup, and in which more than half the operators are of order 2. As this process can clearly be continued until the central has become of order 2 the following theorem has been established: *It is possible to construct a group which admits an automorphism in which more than half the operators correspond to their inverses such that each of the largest subgroups composed of operators which correspond to their inverses under this automorphism is invariant and has an index under the group which exceeds any arbitrarily assigned number.*

Suppose that the index of H under G is an odd number. It results directly from the noted theorems that H is transformed into itself only by its own operators and that s must be commutative with exactly half of the operators of H. The operators which are commutative with s constitute the central of G and the order of the central quotient group of G is twice an odd number. This quotient group must therefore involve a subgroup of index 2 composed of its operators of odd order together with the identity. As the remaining operators of this quotient group are of order 2 its maximal subgroup of odd order must be abelian. It therefore results that G involves a sub-group of index 2 which is the direct product of an abelian group and a group of odd order which is simply isomorphic with the maximal subgroup of odd order contained in the central quotient group. Hence we have proved the following theorem: *If a group G admits an automorphism in which more than half the operators correspond*

to their inverses and if one of the largest subgroups composed of operators which correspond to their inverses under this automorphism is of odd index, then G contains an abelian subgroup of index 2 which is the direct product of an abelian group whose order is this odd index and another abelian group.

From this theorem it results that when the index of H under G is an odd number then G admits a number of conjugate automorphisms in which more than one-half of the operators correspond to their inverses which is equal to this index. In each of these all the operators of the abelian subgroup of index 2 correspond to their inverses and the number of additional operators which correspond to their inverses is equal to the order of the central of G. It therefore results that whenever H is of index $2k + 1$, k being a positive integer, then G admits exactly $2k + 2$ automorphisms in which more than half of its operators correspond to their inverses. One of these is characteristic while the rest form a single set of conjugates under G. A necessary and sufficient condition that the characteristic automorphism is the identical automorphism is that each of the operators in H besides the identity is of order 2. The simplest illustration of this general theorem is furnished by the automorphism of the symmetric group of order 6. It is not difficult to determine the number of distinct groups of order g which belong to the given category. This is clearly the product of the number of abelian groups of order $2k + 1$ and the sum of the numbers of the sets of subgroups of index 2 contained in the abelian groups of order $g/(2k + 1)$ such that each set is composed of all those subgroups which are conjugate under the groups of isomorphisms of the corresponding abelian groups.

LINEAR TRANSFORMATIONS IN HILBERT SPACE. III. OPERATIONAL METHODS AND GROUP THEORY

By M. H. Stone

Department of Mathematics, Harvard University

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In the present note, we wish to outline certain further developments of the theory of transformations in Hilbert space sketched in previous notes.[1] The results which we shall give have a somewhat special and fragmentary character, due to the fact that they are here isolated as the most interesting and novel aspects of the theory in which they appear.

On the basis of the results already outlined, we can lay the foundations of an operational calculus: if T is an arbitrary self-adjoint transformation and E_λ is the corresponding canonical resolution of the identity, we

define $F(T)$ as the transformation whose field comprises those and only those elements f such that $\int_{-\infty}^{+\infty} |F(\lambda)|^2 \, dQ(E_\lambda f)$ converges and which takes f into the element $F(T)f$ such that

$$Q(F(T)f, g) = \int_{-\infty}^{+\infty} F(\lambda) \, dQ(E_\lambda f, g)]$$

for every g in $\mathfrak{H}$. In this definition, the integrals are to be treated as Lebesgue-Stieltjes integrals, and the function $F(\lambda)$ is required to have such attributes as to ensure the possibility of this interpretation; in particular, the definition is always significant for Borel measurable functions. By means of a systematic development of the requisite fundamental theorems, we justify the use of the symbol $F(T)$, which reflects the isomorphism between the algebra of the transformations and that of the functions defining them, and obtain a full-fledged operational calculus suitable for application to other problems. We may mention that a particular case of great interest arises in connection with the differential operator $i\frac{d}{dx}$ in the Hilbert space consisting of all complex-valued Lebesgue measurable functions $f(x)$, $-\infty < x < +\infty$, for which the integral $\int_{-\infty}^{+\infty} |f(x)|^2 \, dx$ exists: the differential operator defines a self-adjoint transformation, and the related operational calculus is the Heaviside calculus. In this instance, our definitions coincide with those given by Wiener.[2]

A first important application of the operational calculus yields a beautiful discussion of the theory of the unitary equivalence of self-adjoint transformations, a theory elaborated for bounded transformations in an essentially identical but less perspicuous form by Hellinger[3] and Hahn.[4] We shall not give details here.

A second application, to a field in which few results have been published hitherto, concerns group-theoretic questions of fundamental importance in the quantum mechanics. Among these questions is the study of the unitary representations of the group of translations of a straight line into itself. The solution of this problem is embodied in the following statement:

THEOREM. If U_τ, $-\infty < \tau < +\infty$ is a group of unitary transformations in $\mathfrak{H}$ such that

$$U_0 = I, \quad U_{-\tau} = U_\tau^{-1}, \quad U_{\sigma+\tau} = U_\sigma U_\tau,$$

then there exists a unique self-adjoint transformation T with the canonical resolution of the identity E_λ, such that $Q(U_\tau f, g) = \int_{-\infty}^{+\infty} e^{i\tau\lambda} dQ(E_\lambda f, g)$

while $\frac{U_\tau - U_0}{\tau} f \longrightarrow iTf$ when $\tau \longrightarrow 0$ and f is in the field of T; the transformation iT may be called the generating infinitesimal transformation of the group. Conversely, when T is a given self-adjoint transformation, there exists a one-parameter group of unitary transformations of the type described above which has iT as its generating infinitesimal transformation.[5]

The device upon which the demonstration of this theorem depends is a transition from the group U_τ to the resolvent R_l of T, by means of Fourier analysis: we form the function $\psi(\tau; l) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\lambda\tau}\frac{1}{\lambda - l}\,d\lambda$, and then study the integral $\int_{-\infty}^{+\infty} Q(U_\tau f, g)\psi(\tau; l)d\tau$, showing that it defines a transformation R_l which is the resolvent of a self-adjoint transformation T.

A second question of group theory, to which we can apply the operational calculus, is raised by the Heisenberg permutation relations connecting the self-adjoint transformations P_k, Q_k, $k = 1, \ldots, n$. For convenience, we write these relations in the form

$$P_kQ_l - Q_lP_k = i\delta_{kl}I,\ P_kP_l - P_lP_k = 0,\ Q_kQ_l - Q_lQ_k = 0,\ k, l = 1, \ldots, n.$$

In quantum mechanics, these transformations refer essentially to the coördinates and momenta of a dynamical system of n degrees of freedom. The content of these permutation relations must be made precise by expressing them in terms of the one parameter groups of unitary transformations $U_\tau^{(k)}$ and $V_\tau^{(k)}$ generated by iP_k and iQ_k, respectively. We have $U_\sigma^{(k)}\ V_\tau^{(k)} = e^{-i\sigma\tau}\ V_\tau^{(k)}U_\sigma^{(k)}$, and $U_\sigma^{(k)}\ V_\sigma^{(l)} = V_\tau^{(l)}U_\sigma^{(k)}$ when k and l are different, for $k, l = 1, \ldots, n$; we have also $U_\sigma^{(k)}\ U_\tau^{(l)} = U_\tau^{(l)}\ U_\sigma^{(k)}$ and $V_\sigma^{(k)}\ V_\tau^{(l)} = V_\tau^{(l)}\ V_\sigma^{(k)}$ for $k, l = 1, \ldots, n$. We prove the following theorem:

THEOREM. If the family of transformations $U_{\sigma_1}^{(1)} \ldots U_{\sigma_n}^{(n)}\ V_{\tau_1}^{(1)} \ldots V_{\tau_n}^{(n)}$ is irreducible in $\mathfrak{H}$, then there exists a one-to-one linear isometric correspondence or transformation S which takes $\mathfrak{H}$ into $\mathfrak{H}_n$, the space of all complex-valued Lebesgue-measurable functions $f(x_1, \ldots, x_n)$, $-\infty < x_1 < +\infty, \ldots, -\infty < x_n < +\infty$, for which the integral $\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} |f(x_1, \ldots, x_n)|^2 dx_1 \ldots dx_n$ exists, such that

$$SP_kS^{-1}f(x_1, \ldots, x_n) = i\frac{\partial}{\partial x_k}f,\ SQ_kS^{-1}f(x_1, \ldots, x_n) = x_kf.$$

The proof of this theorem depends upon a consideration of the transformations

$$T_k = Q_k P_k Q_k - P_k, \quad S_k = (Q_k - i)/(Q_k + i)$$

for $k = 1, \ldots, n$, and their mutual relations. The principal difficulty lies in showing that the transformations T_k are self-adjoint. The determination of their spectra, under the hypothesis of irreducibility, can then be effected and leads easily to the construction of the desired transformation S by means of a device previously employed by J. v. Neumann in a rather different connection.[6]

The significance of the last two theorems for the quantum mechanics has been pointed out by Weyl, who made no attempt to prove them.[7] We shall add one or two remarks concerning the last theorem. If $\mathfrak{H}$ is the representation space for the states of a dynamical system of n degrees of freedom, the postulate of irreducibility imposed in addition to the Heisenberg permutation relations has the following physical significance: if the family of transformations considered were reducible, it would be reduced by each of two mutually perpendicular or orthogonal closed linear manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$, both proper subsets of $\mathfrak{H}$; these manifolds would reduce any transformation expressible as a function of the transformations P_d, Q_k, $k = 1, \ldots, n$; hence no physically realizable perturbation could take a state representable in $\mathfrak{M}_1$ into a state representable in $\mathfrak{M}_2$, or vice versa; our postulate of irreducibility avoids such a degeneracy into two physically distinct systems. We may also remark that the result of the theorem leads to the following important fact: if we have two sets of canonical coördinates P_k, Q_k and P_k', Q_k', satisfying the Heisenberg relations and the postulate of irreducibility, we can obtain a unitary transformation of $\mathfrak{H}$ into itself, which we may call U, with the property that $P_k' = UP_kU^{-1}$, $Q_k' = UQ_kU^{-1}$, for $k = 1, \ldots, n$. The converse of this assertion is trivial.

[1] *Proc. Nat. Acad. Sci.*, **15** (1929), 198–200, 423–425.

[2] Wiener, *Mathematische Annalen*, **95** (1925), 557–584.

[3] Hellinger, *J. Mathematik*, **136** (1909), 210–271.

[4] Hahn, *Monatshefte Mathematik und Physik*, **23** (1912),161–224.

[5] Presented to the American Mathematical Society, December 27, 1928; see *Bull. Am. Math. Soc.*, **35** (1929), 167.

[6] J. v. Neumann, *Mathematische Annalen*, **102** (1929), 49–131, particularly Anhang 4; this is the paper of v. Neumann referred to in footnote 6 of our preceding note.

[7] Weyl, *Gruppentheorie und Quantenmechanik*, Leipzig, **1928**, §§. 14, 15, 18, 45, 46.